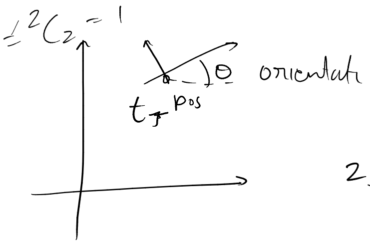
3D Coordinate Transformations

$$R(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}_{2\times 2} = 1\text{ DOF} = \theta \stackrel{?}{=} {}^2C_2 = 1$$



$$T = \begin{bmatrix} R(\theta)_{2\times 2} & t_{2\times 1} \\ 0^T_{2\times 1} & 1 \end{bmatrix}_{3\times 3} = 3\text{ DOF}$$

θ — 1 DOF orient

t_j — 2 DOF – pos = trans

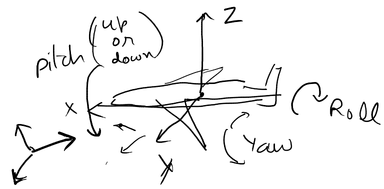
For a 3D car how many DOF?

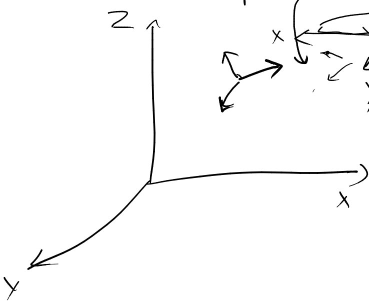
translation = 3 DOF x – y, –

Rotation $\stackrel{?}{=}$ 1 DOF per axis = n in n-D

$\stackrel{?}{=}$ 1 DOF per 2D plane

$= {}^nC_2 = {}^3C_2 = \frac{3\times 2\times 1}{(2\times 1)} = 3$





4D space DOF for rotation = ${}^4C_2$ = 12 ?

DOF " translation = 4

Think about it

3D case

How do we represent. Rotation.

Rotation matrix

$$\underbrace{R \in \mathbb{R}^{3\times 3}_{\text{Real}}, \quad R^T R = R R^T = I, \quad \det(R) = 1}_{R \in SO(3)}$$

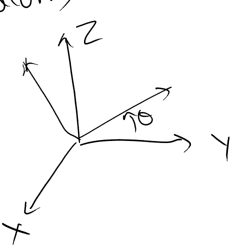
2D

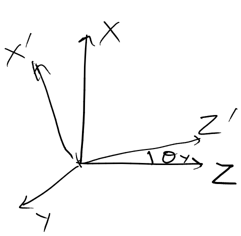
$R(\theta)$ angle of rotation

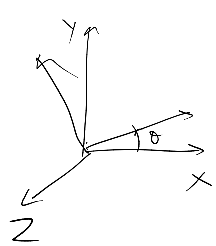
|Roll, pitch, yaw| rotation along axis in a a sequence

$$R_x(\theta_x) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_x & -\sin\theta_x \\ 0 & \sin\theta_x & \cos\theta_x \end{bmatrix}, \quad R_y(\theta_y) = \begin{bmatrix} \cos\theta_y & 0 & \sin\theta_y \\ 0 & 1 & 0 \\ -\sin\theta_y & 0 & \cos\theta_y \end{bmatrix}, \quad R_z(\theta_z) = \begin{bmatrix} \cos\theta_z & -\sin\theta_z & 0 \\ \sin\theta_z & \cos\theta_z & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Principal Rotations







Roll pitch Yaw

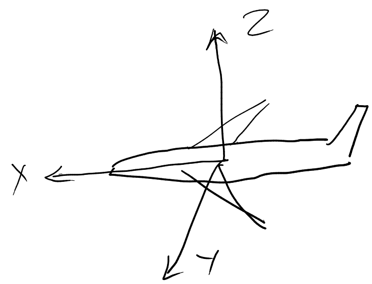
$$R = \underbrace{R_z(\theta_z)}_{\text{Yaw}} \underbrace{R_y(\theta_y)}_{\text{pitch}} \underbrace{R_x(\theta_x)}_{\text{Roll}} P_R$$

$R = R_{XYZ}$

$R = XYX \quad ZYX \quad$ 12 possibilities

Euler angles

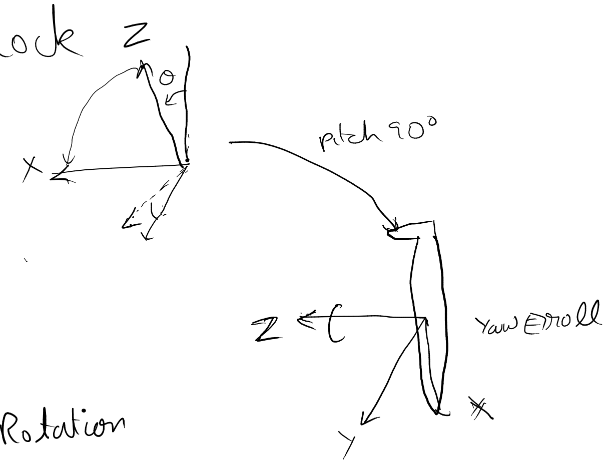
Euler angle problem: Gimbal lock



Roll = 10° →

pitch = 90°

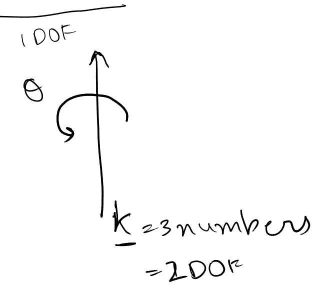
Yaw = 10°



Euler angle → Rotation

Avoid euler angle

Axis - angle representation

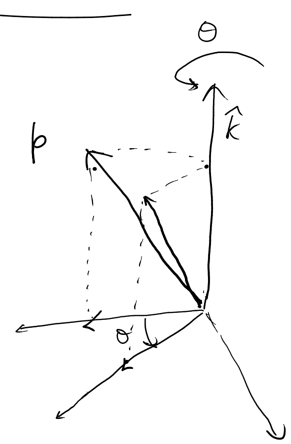


$\theta = \|k\| \qquad \hat{k} = \frac{k}{\|k\|}$

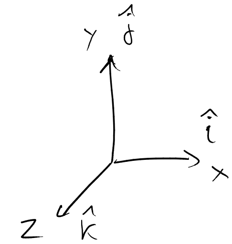
$\theta = 0 \Rightarrow$ no rotation

How to get the rotation Matrix

Rodrigues formula



3D



cross product

$$a \times b = \begin{vmatrix} \hat{i} & -\hat{j} & \hat{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$$

$$= (a_y b_z - b_y a_z)\hat{i} - (a_x b_z - b_x a_z)j + (a_x b_y - b_x a_y)\hat{k}$$

$\hat{j} \times \hat{i} = -\hat{k} \quad . \quad \hat{i} \times \hat{j} = \hat{k}$

$\hat{j} \times \hat{k} = \hat{i}$

$\hat{k} \times \hat{i} = \hat{j}$

$$\underline{a} \times \underline{b} = [a]_\times b$$

$$= \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix} \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix}$$

cross-product matrix

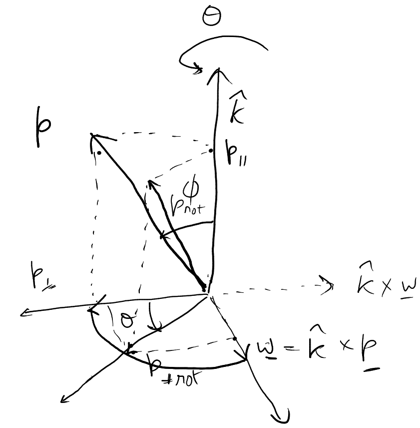
$\boxed{\text{Lie algebra}}$

postpone

How to differentiate w.r.t. rotation

$$\frac{\partial}{\partial \omega} f(R)$$

$$p_{rot} = p_{\parallel} + p_{\perp rot}$$

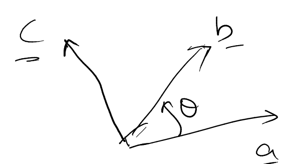


$$p_{\parallel} = (p^T \hat{k})\hat{k}$$

$$p_{\perp} = p - p_{\parallel}$$

$w \perp p$ and $w \perp \hat{k}$

unique $\underline{c} = \underline{a} \times \underline{b}$ $\quad \underline{c} \perp \underline{a}$ and $\underline{c} \perp \underline{b}$



$$\|c\| = \|a\| \|b\| \sin\theta$$

$$\underline{w} = \hat{k} \times p = [k]_\times p \qquad \|\underline{w}\| = \|p\| \sin\phi \quad , \; p_{\perp} = \|p\| \sin\phi$$

$$p_{\perp} = -\hat{k} \times \underline{w} = -[k]_\times \{[k]_\times p\} = -[k]_\times^2 \underline{p}$$

$$p_{\perp rot} = \alpha \hat{p}_{\perp} + \beta \hat{w}$$

$$= \|p_{\perp}\| \cos\theta \, \hat{p}_{\perp} + \|p_{\perp}\| \sin\theta \, \hat{w}$$

$$= p_{\perp} \cos\theta + \underline{w} \sin\theta$$

$$p_{\perp rot} = -[k]_\times^2 p \cos\theta + [k]_\times p \sin\theta$$

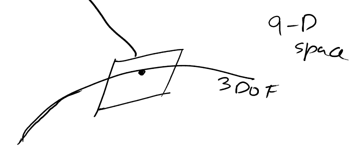
$$p_{rot} = p_{\parallel} + p_{\perp rot}$$

$$= (p - p_{\perp}) + p_{\perp rot}$$

$$= p + \lfloor k \rfloor_{\times}^{2} p - \lfloor k \rfloor_{\times}^{2} p \cos\theta + \lfloor k \rfloor_{\times} p \sin\theta$$

$$= \underbrace{\left[I + \lfloor k \rfloor_{\times} \sin\theta + \lfloor k \rfloor_{\times}^{2} (1 - \cos\theta) \right]}_{R(\hat{k},\theta)} \underline{p}$$

$$\underbrace{R(\hat{k}, \theta)}_{SO(3)} = \exp(\underbrace{\theta \lfloor k \rfloor_{\times}}_{\text{Exp map}}) \longleftrightarrow \text{Lie algebra} \qquad \theta \lfloor k \rfloor_{\times}$$



Lie group [+ is [group / differentiable manifold]]

Axis - angle representation ✓

trignometric operations { $\sin\theta$, $\cos\theta$ }

Quaternions