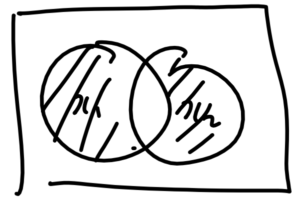
8 Karnaugh maps = (truth table + Venn diagram)

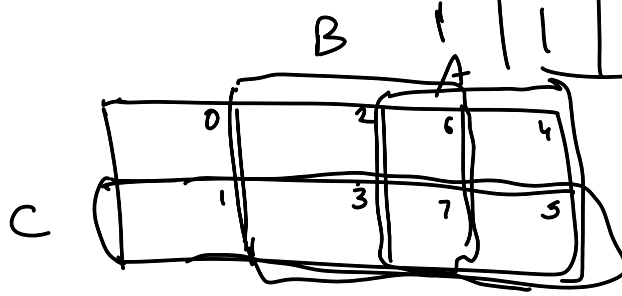
XOR gate

x_1	x_2	f
0	0	0
0	1	1
1	0	1
1	1	0



	$\bar{x}_2$	x_2
$\bar{x}_1$	0	1
x_1	1	0

x_2 \ x_1	0	1
0	0	1
1	1	0



8.1 Two input K-maps

B \ A	0	1
0	m_0	m_2
1	m_1	m_3

8.2 Three input K-maps

0 1 3 → 2

C \ AB	00	01	11	10
0	m_0	m_2	m_6	m_4
1	m_1	m_3	m_7	m_5

8.3 Four input K-maps

CD \ AB	00	01	11	10
00	m_0	m_4	m_{12}	m_8
01	m_1	m_5	m_{13}	m_9
11	m_3	m_7	m_{15}	m_{11}
10	m_2	m_6	m_{14}	m_{10}

8.4 Five input K-maps

A = 0

DE \ BC	00	01	11	10
00	m_0	m_4	m_{12}	m_8
01	m_1	m_5	m_{13}	m_9
11	m_3	m_7	m_{15}	m_{11}
10	m_2	m_6	m_{14}	m_{10}

A = 1

DE \ BC	00	01	11	10
00	m_{16}	m_{20}	m_{28}	m_{24}
01	m_{17}	m_{21}	m_{29}	m_{25}
11	m_{19}	m_{23}	m_{31}	m_{27}
10	m_{18}	m_{22}	m_{30}	m_{26}

9 More Gates and notations summary

<table>
<tr><th>Name</th><th>C/Verilog</th><th>Boolean expr.</th><th>Truth Table</th><th>(ANSI) symbol</th><th>K-map</th></tr>
<tr><td>NAND Gate</td><td><code>Q = ~(x1 & x2)</code></td><td>$Q = \overline{x_1 \cdot x_2} = \overline{x_1 x_2}$</td><td>
<table>
<tr><th>x_1</th><th>x_2</th><th>$\overline{x_1 \cdot x_2}$</th></tr>
<tr><td>0</td><td>0</td><td>1</td></tr>
<tr><td>0</td><td>1</td><td>1</td></tr>
<tr><td>1</td><td>0</td><td>1</td></tr>
<tr><td>1</td><td>1</td><td>0</td></tr>
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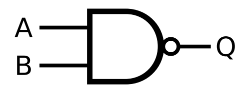
</td><td>
</td><td>
<table>
<tr><th>B \ A</th><th>0</th><th>1</th></tr>
<tr><td>0</td><td>1</td><td>1</td></tr>
<tr><td>1</td><td>1</td><td>0</td></tr>
</table>
</td></tr>
<tr><td>NOR Gate</td><td><code>Q = ~(x1 | x2)</code></td><td>$Q = \overline{x_1 + x_2}$</td><td>
<table>
<tr><th>x_1</th><th>x_2</th><th>$\overline{x_1 + x_2}$</th></tr>
<tr><td>0</td><td>0</td><td>1</td></tr>
<tr><td>0</td><td>1</td><td>0</td></tr>
<tr><td>1</td><td>0</td><td>0</td></tr>
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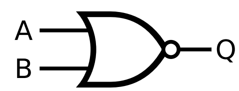
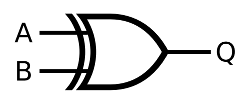
</td><td>
</td><td>
<table>
<tr><th>B \ A</th><th>0</th><th>1</th></tr>
<tr><td>0</td><td>1</td><td>0</td></tr>
<tr><td>1</td><td>0</td><td>1</td></tr>
</table>
</td></tr>
<tr><td>XOR Gate</td><td><code>Q = x1 ^ x2</code></td><td>$Q = x_1 \oplus x_2$</td><td>
<table>
<tr><th>x_1</th><th>x_2</th><th>$x_1 \oplus x_2$</th></tr>
<tr><td>0</td><td>0</td><td>0</td></tr>
<tr><td>0</td><td>1</td><td>1</td></tr>
<tr><td>1</td><td>0</td><td>1</td></tr>
<tr><td>1</td><td>1</td><td>0</td></tr>
</table>
</td><td>
</td><td>
<table>
<tr><th>B \ A</th><th>0</th><th>1</th></tr>
<tr><td>0</td><td>0</td><td>1</td></tr>
<tr><td>1</td><td>1</td><td>0</td></tr>
</table>
</td></tr>
<tr><td>XNOR Gate</td><td><code>Q = ~(x1 ^ x2)</code></td><td>$Q = \overline{x_1 \oplus x_2}$</td><td>
<table>
<tr><th>x_1</th><th>x_2</th><th>$\overline{x_1 \oplus x_2}$</th></tr>
<tr><td>0</td><td>0</td><td>1</td></tr>
<tr><td>0</td><td>1</td><td>0</td></tr>
<tr><td>1</td><td>0</td><td>0</td></tr>
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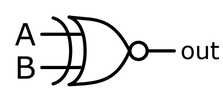
</td><td>
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<table>
<tr><th>B \ A</th><th>0</th><th>1</th></tr>
<tr><td>0</td><td>1</td><td>0</td></tr>
<tr><td>1</td><td>0</td><td>1</td></tr>
</table>
</td></tr>
</table>

NAND gate

A	B	f
0	0	1
0	1	1
1	0	1
1	1	0

K - map

B \ A	$\bar{A}$ (A = 0)	A (A = 1)
$\bar{B}$ (B = 0)	1	1
B (B = 1)	1	0

NOR gate

$y = A + B$

OR → NOT

A	B	y	f
0	0	0	1
0	1	1	0
1	0	1	0
1	1	1	0

K-map

B ↓ \ A →	0	1
0	1	0
1	0	0

$\overrightarrow{A, B, C, D}$

Example 10. *Convert the following Boolean expression into a K-map.* $f = \overline{A\bar{B} + CD}$

$f = m_1 + m_5 + m_{13}$

$f =$

	$\bar{B}$	B	B (A)	$\bar{B}$ (A)	
	^{0} 0	^{4} 0	^{12} 0	^{8} 0	
	^{1} 1	^{5} 1	^{13} 1	^{9} 0	D
C	^{3} 0	^{7} 0	^{15} 0	^{11} 0	D
C	^{2} 0	^{6} 0	^{14} 0	^{10} 0	

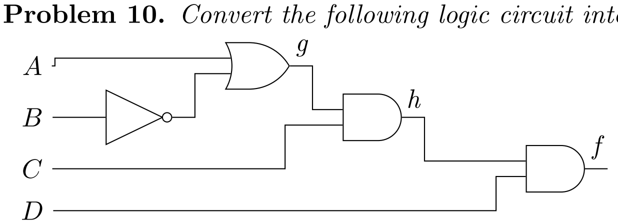
$g_1 = A\bar{B}$

$g_2 = g_1 + C$

$g_3 = \overline{g_2}$

$f = g_3 \cdot D$

Problem 10. *Convert the following logic circuit into a K-map.*



10 Boolean Algebra

10.1 Axioms of Boolean algebra

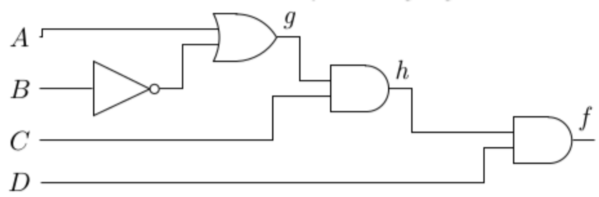
1. $0 \cdot 0 = 0$
2. $1 + 1 = 1$

$0 \cdot 0 = 0$

$0 \cdot 1 = 0$

$1 \cdot 0 = 0$

$1 \cdot 1 = 1$



$g =$

$g = A + \overline{B}$

AB →, CD ↓

CD \ AB	00	01	11	10
00	1	0	1	1
01	1	0	1	1
11	1	0	1	1
10	1	0	1	1

A B C D

AB →, CD

CD \ AB				
00	0	0	0	0
01	0	0	0	0
11	1	0	1	1
10	1	0	1	1

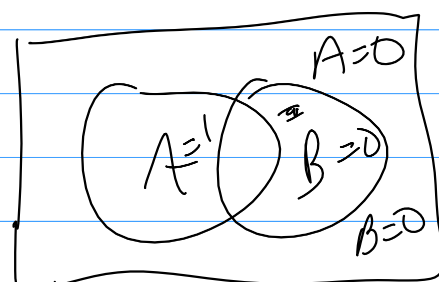
$h = g \cdot C$

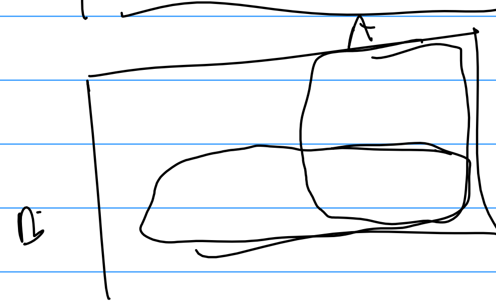
1	·	0	=	0
1		0		0
1		1		1
1		1		1

$A =$

AB

	0	0	1	1
	0	0	1	1
C	0	0	1	1
	0	0	1	1





C AB

A

0	0	0	0
0	0	0	0
1	1	1	1
0	0	1	1

C

B

$f =$

A

0	0	0	0
0	0	0	0
1	0	1	1
0	0	0	0

C D

B

$f = h \cdot D$

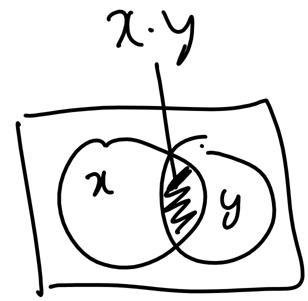
$= (A + \overline{B})\, C \cdot D$

Duality exists in axiom

3. $1 \cdot 1 = 1$
4. $0 + 0 = 0$
5. $0 \cdot 1 = 1 \cdot 0 = 0$
6. $\bar{0} = 1$
7. $\bar{1} = 0$
8. $x = 0$ if $x \neq 1$
9. $x = 1$ if $x \neq 0$

$\bar{0} = 1$

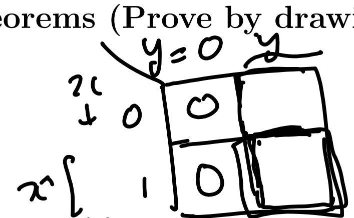
$\bar{1} = 0$



10.2 Single variable theorems (Prove by drawing K-maps)

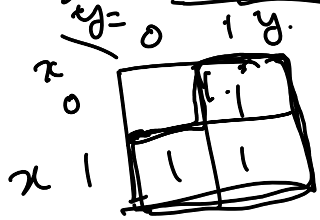
Duality
$\cdot \to +$
$0 \to 1$
$+ \to \cdot$
$1 \to 0$

1. $x \cdot 0 = 0$



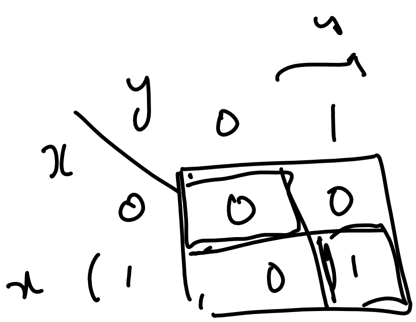
2. $x + 1 = 1$

Th 2 is the dual of Th. 1



3. $x \cdot 1 = x$

4. $x + 0 = x$



x = 0

5. $x \cdot x = x$

6. $x + x = x$

7. $x \cdot \bar{x} = 0$

8. $x + \bar{x} = 1$

9. $\bar{\bar{x}} = x$

Remark 2 (Duality)**.** *Swap + with · and 0 with 1 to get another theorem*

10.3 Two and three variable properties (Prove by K-maps)

1. Commutative: $x \cdot y = y \cdot x$, $x + y = y + x$

2. Associative: $x \cdot (y \cdot z) = (x \cdot y) \cdot z$, $x + (y + z) = (x + y) + z$

$x \cdot y \cdot z$

Duals

3. Distributive: $x \cdot (y + z) = x \cdot y + x \cdot z$, $\boxed{y + x \cdot z}$ $x \cancel{+ y} \cdot z = (x + y) \cdot (y + z)$

$x + y \cdot z = (x + y) \cdot (x + z)$

$\text{RHS} = (x+y)\cdot(y+z) = x \cdot (y+z) + y \cdot (y+z)$

$= x \cdot y + x \cdot z + y \cdot y + y \cdot z$

$= x \cdot y + x \cdot z + y + y \cdot z = y \cdot (x + 1 + z) + x \cdot z$

$= y(1 + z) + x \cdot z$

$= y \cdot 1 + x \cdot z = y + x \cdot z$

4. Absorption: $x + x \cdot y = x$, $x \cdot (x + y) = x$

$LHS = x + y \cdot z$

$z \backslash xy$				
	0	0	1	1
z	0	1	1	1

$RHS = (x+y) \cdot (y+z)$

$x+y$

$z \backslash xy$				
	0	1	1	1
z	0	1	1	1

$y+z =$

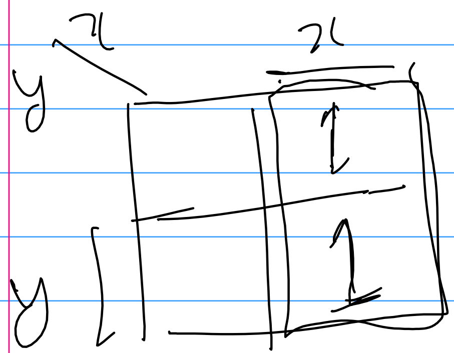
$z \backslash xy$				
	0	1	1	0
z	1	1	1	1

$z \backslash xy$				
	0	1	1	0
z	0	1	1	1

$y + x \cdot z$

$$x + y \cdot x = x$$

$$x \cdot (y + x) = x$$



$$\text{LHS} = x + y \cdot x = x(1 + y)$$

$$= x \cdot 1$$

$$= x = \text{RHS}$$

$$\text{LHS} = x(y + x) = x \cdot y + x \cdot x$$

$$= x \cdot y + x$$

$$= x(y + 1) = x \cdot 1 = x = \text{RHS}$$

5. Combining: $\cancel{x \cdot y + x \cdot \bar{y}}$, $(x + y) \cdot (x + \bar{y}) = x$

$x \cdot y + x \cdot \bar{y} = x$

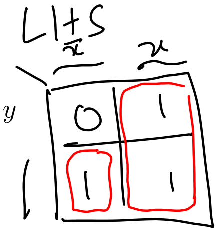
$$\boxed{\text{6. DeMorgan's theorem: } \overline{x \cdot y} = \bar{x} + \bar{y},\ \overline{x + y} = \bar{x} \cdot \bar{y}.}$$

7. Concensus:

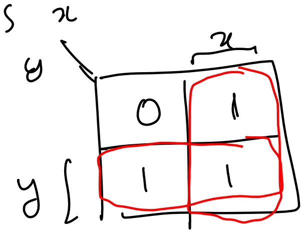
Duals

(a) $x + \underline{\bar{x} \cdot y} = x + y$

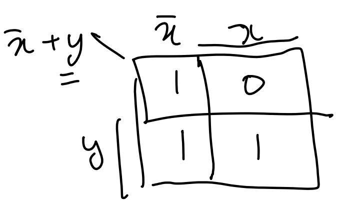
LHS



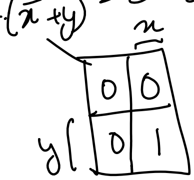
RHS



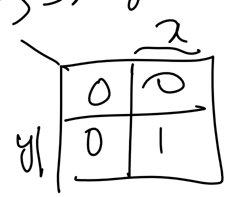
(b) $x \cdot (\bar{x} + y) = x \cdot y$



$x \cdot (\bar{x} + y) = \text{LHS}$

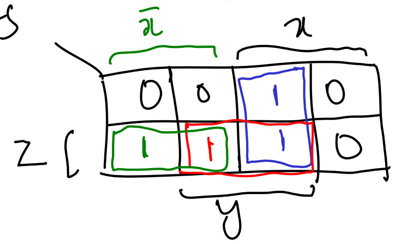


$\text{RHS} = x \cdot y$

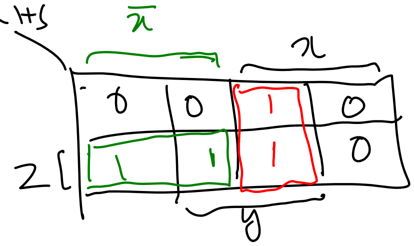


(c) $x \cdot y + \boxed{y \cdot z} + \bar{x} \cdot z = x \cdot y + \bar{x} \cdot z$

LHS



RHS



Prove this using K-maps $\longrightarrow$ (d) $(x + y) \cdot (y + z) \cdot (\bar{x} + z) = (x + y) \cdot (\bar{x} + z)$

$$(x+y)\cdot(y+z)\cdot(\bar{x}+z) = (x+y)\cdot(\bar{x}+z)$$

$x+y$ ($\bar{x}$, x)

$\bar{z}$	0	1	1	1
z	0	1	1	1

$\bar{y}$ y $\bar{y}$

$y+z$ (x)

	0	1	1	0
z	1	1	1	1

y

$\bar{x}+z$ (x)

	1	1	0	0
z	1	1	1	1

y

$LHS = (x+y)(y+z)(\bar{x}+z)$

x

	0	1	0	0
z	0	1	1	1

y

$RHS = (x+y)(\bar{x}+z)$

$x+y$ (x)

	0	1	1	1
z	0	1	1	1

y

$\bar{x}+z$ (x)

	1	1	0	0
z	1	1	1	1

y

$RHS = (x+y)(\bar{x}+z)$

x

	0	1	0	0
z	0	1	1	1

y

De Morgan' Theorem

$$\overline{x \cdot y} = \bar{x} + \bar{y}$$

$$\overline{x \cdot y \cdot z} = \bar{x} + \bar{y} + \bar{z}$$

$$\overline{(x + y + z)} = \bar{x} \cdot \bar{y} \cdot \bar{z}$$

LHS = $\overline{x \cdot y}$

$x \cdot y =$

$y \backslash x$		x
	0	0
y	0	1

$\rightarrow$

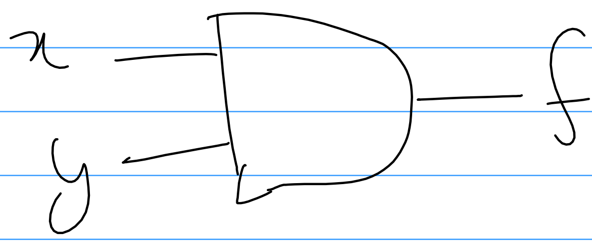
$\overline{x \cdot y}$

$y \backslash x$		x
	1	1
y	1	0

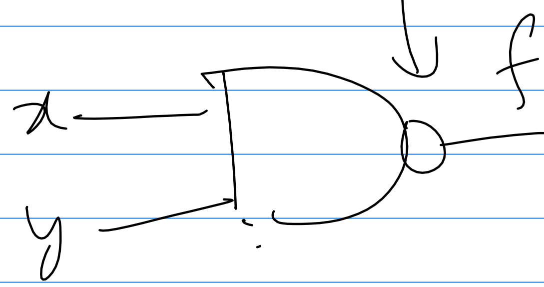
RHS = $\bar{x} + \bar{y}$

$y \backslash x$	$\bar{x}$	x
$\bar{y}$	1	1
y	1	0

De Morgan Theorem as Bubble pushing



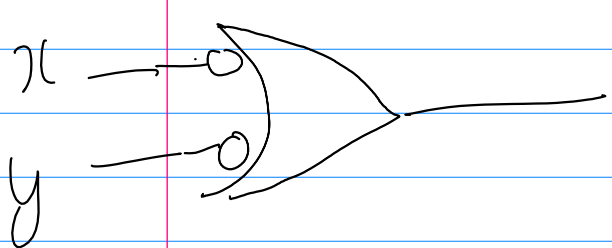
NOT gate

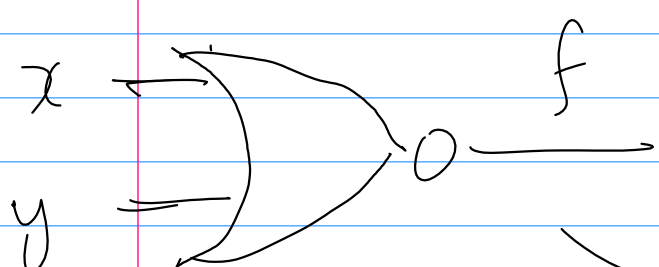


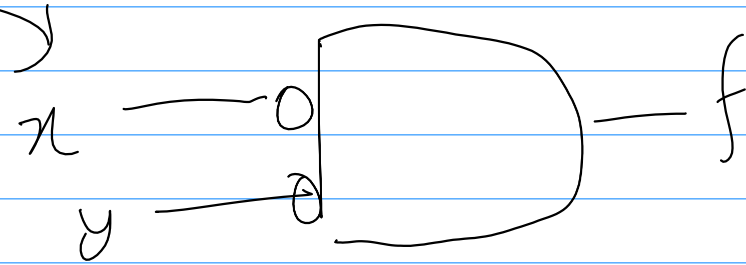
NAND

$$f = \overline{x \cdot y} = \bar{x} + \bar{y}$$

Bubble pushed through AND gate

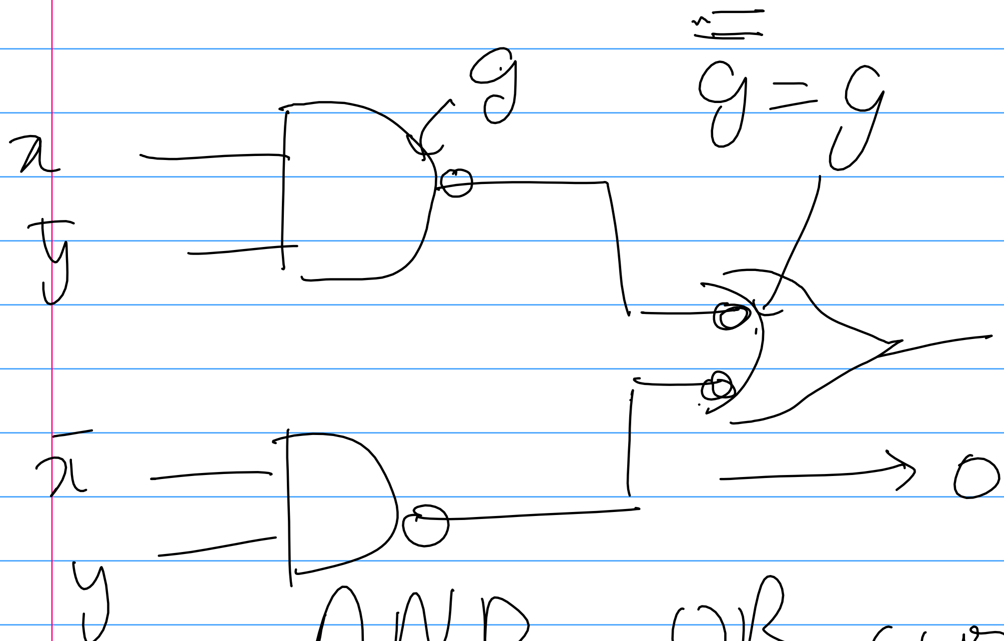




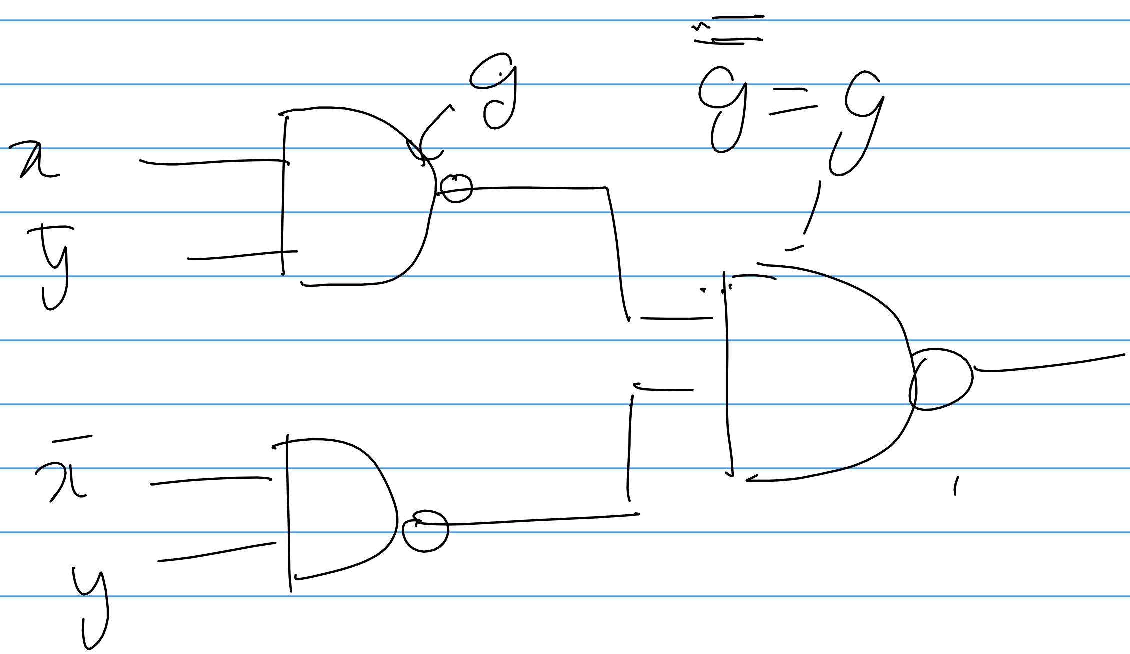


Sum of products form

$$f = x\bar{y} + \bar{x}y$$



AND-OR circuit



NAND-NAND circuit

$$LHS = x \cdot y + x \cdot \bar{y}$$
$$= x \cdot \underbrace{(y + \bar{y})}$$
$$= x \cdot 1$$
$$= x \cdot = RHS$$

Dual

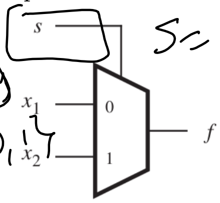
$$LHS = (x + y)(x + \bar{y})$$
$$= \underbrace{x \cdot x} + x \cdot \bar{y} + y \cdot x + y \cdot \bar{y}$$
$$= x + x \cdot \bar{y} + y \cdot x + \underbrace{y \cdot \bar{y}}_{0}$$
$$= x + x\bar{y} + y \cdot x + 0$$
$$= x + x(y + \bar{y})$$
$$= x + x \cdot 1$$
$$= x + x$$
$$= x$$

Example 11 (Multiplexer). *Multiplexer is a circuit used to select one of the input lines x_1 and x_2 based only select input s. When $s = 0$, x_1 is selected, x_2 is selected otherwise. Find a boolean expression and a circuit for multiplexer*

$s = 0$

$x_1 \in \{0, 1\}$

$x_2 \in \{0, 1\}$



s	$f(s, x_1, x_2)$
0	x_1
1	x_2

s	x_1	x_2	f
0	0	0	0
0	0	1	0
0	1	0	1
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	0
1	1	1	1

Example 12. *Simplify $f = \bar{A}\bar{B}\bar{C} + A\bar{B}\bar{C} + A\bar{B}\bar{C}$ using boolean algebra*

$$f = \underbrace{\bar{A}\bar{B}\bar{C}}_{m_0} + \underbrace{A\bar{B}\bar{C}}_{m_4} + \underbrace{AB\bar{C}}_{m_6}$$

$$= \underbrace{(\bar{A} + A)}_{1}\bar{B}\bar{C} + A\underbrace{(\bar{B} + B)}_{1}\bar{C}$$

$$= \bar{B}\bar{C} + A\bar{C}$$

Example 13. *Simplify $f = \bar{A}\bar{A}\bar{C} + \bar{A}\bar{B}C$ using K-maps.*

f			A	
	1 (0)	0 (2)	1 (6)	1 (4)
C	0 (1)	0 (3)	0 (7)	0 (5)
	$\bar{B}$	B	B	$\bar{B}$

$f = A\bar{C} + \bar{B}\bar{C}$

$$x + x = x$$
$$x \cdot x = x$$

$$f = \underbrace{\bar{A}\bar{B}\bar{C}}_{m_0} + \underbrace{A\bar{B}\bar{C}}_{m_4} + \underbrace{AB\bar{C}}_{m_6}$$

$$= \bar{A}\bar{B}\bar{C} + A\bar{B}\bar{C} + A\bar{B}\bar{C} + AB\bar{C}$$

$$= \underbrace{(\bar{A} + A)}_{1}\bar{B}\bar{C} + A\underbrace{(\bar{B} + B)}_{1}\bar{C}$$

$$= \bar{B}\bar{C} + A\bar{C}$$

$$x + \bar{x} = 1$$
$$x \cdot \bar{x} = 0$$

s, x_1, x_2

	S	x_1	x_2	f
0	0	0	0	0
1	0	0	1	0
2	0	1	0	1
3	0	1	1	1
4	1	0	0	0
5	1	0	1	1
6	1	1	0	0
7	1	1	1	1

$S x_1 \rightarrow$, $x_2 \downarrow$

	00	01	11	10
$\bar{x}_2$	0 (0)	1 (2)	0 (6)	0 (4)
x_2	0 (1)	1 (3)	1 (7)	1 (5)

S over columns 6, 4; x_1 under columns 3, 7.

$S x_1 \rightarrow$ 00 01 10 11, $x_2 \downarrow$

	00	01	10	11

$x_1 x_2 \rightarrow$, $S \downarrow$

	0	1		2
S				

x_1, x_2

x_1	x_2	S	f

---

f

$S x_1 \rightarrow$, $x_2 \downarrow$

	$\bar{S}$		S	
	0 (0)	1 (2)	0 (6)	0 (4)
x_2	0 (1)	1 (3)	1 (7)	1 (5)

$\bar{S}$, x_1

✓ $f = m_2 + m_3 + m_5 + m_7$

✓ $f = x_1 \cdot \bar{S}$

$+ x_2 x_1 \leftarrow$ unnecessary

$+ x_2 \cdot S$

$f = x_1 \cdot \bar{S} + x_2 S$