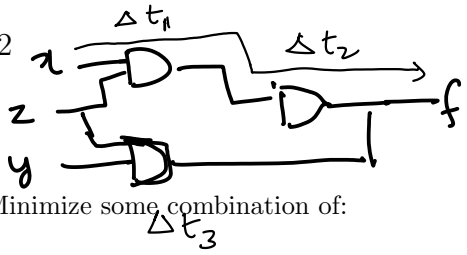
# Logic minimization: Minimum-cost circuits

Vikas Dhiman for ECE275

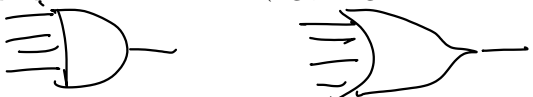
September 19, 2022




## 1 Logic minimization

A general optimization criteria for multi-level logic are to Minimize some combination of:

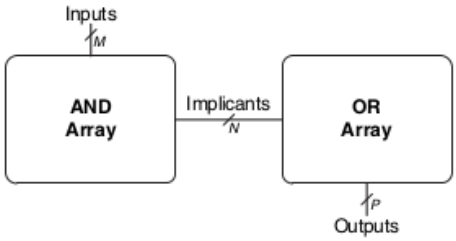
1. Area occupied by the logic gates and interconnect;
2. the Critical Path Delay of the longest path through the logic;
3. the Degree of Testability of the circuit, measured in terms of the percentage of faults covered by a specified set of test vectors, for an appropriate fault model (Eg., single stuck faults, multiple stuck faults, etc.);
4. Power consumed by the logic gates.

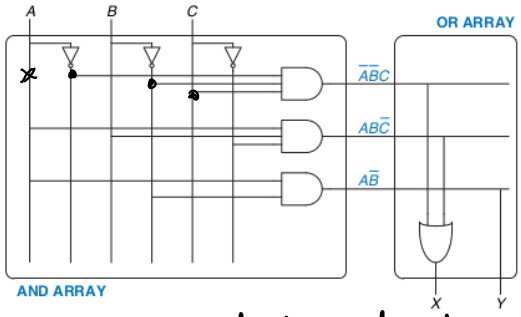


In this course, we will start with two-level multi-input circuits and a criteria based on the number of gates/transistors/diodes.

## 2 Programmable Logic Arrays (PLA)

Two level circuit




sum of products form

## 3 Two-level circuits

The cost that we are going to consider in this class depend upon:

1. Number of gates.
2. Number of input to the gates.

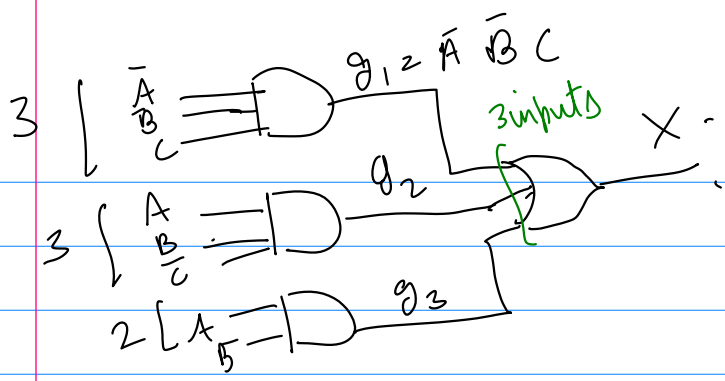
More gates need more transistors, more area on the chip. More-inputs the gate need more transistors within each gate. Number of gate inputs can be considered secondary criterion to the number of gates.

**Example 1.** *Find the cost of the following Boolean expression* $X = \bar{A}\bar{B}C + AB\bar{C} + A\bar{B}$.

1 and + 1 and + 1 and = 3 and gates } 4 gates

**Problem 1.** *Find the cost of the following Boolean expression* $X = A\bar{B}C + \bar{A}B\bar{C} + \bar{B}C$.

= 1 OR gate




$X = g_1 + g_2 + g_3$

4 gates

Total Number of inputs

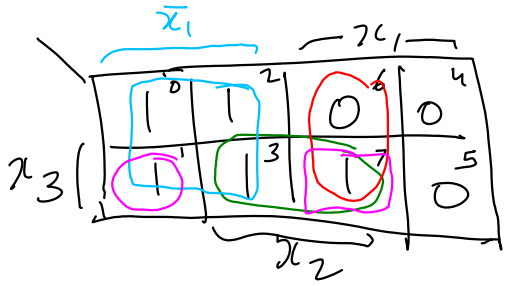
$= \underbrace{3+3+2}_{\text{AND}} + \underbrace{3}_{\text{OR}}$

$= 8+3 = 11$

Total cost $= 4 + 11 = 15$

OR = Union of regions

$f = m_0 + m_1 + m_2 + m_3 + m_7$




## 4 Terminology for K-maps

Running Example: $f = \sum m(0,1,2,3,7) = \bar{x}_1 + x_1 x_2 x_3$.

**Literal** A single variable or its complement. Example: $\bar{x}, x_1, x_2, x_3$

**Implicant** A product term which is true for a function. All minterms are implicants. Example: $x_1x_2x_3$, $\bar{x}_1$, $m_0 = \bar{x}_1\bar{x}_2\bar{x}_3$, $\bar{x}_1x_3$, $\bar{x}_1\bar{x}_3$.

Product Term

$x_1$, $x_2$ ×  $x_2x_3$ ✓  $\bar{x}_1$ ✓

**Prime Implicant** An implicant that cannot be combined into fewer literals. Example: $\bar{x}_1, x_2x_3$.

**Essential Prime Implicant** An implicant that cannot be combined into fewer literals. Example: $x_2x_3$.

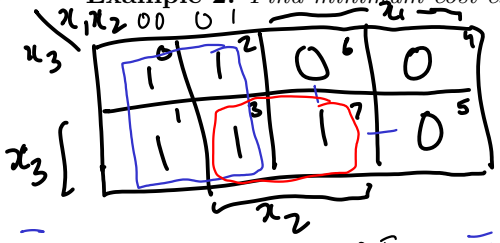
$m_1 = \bar{x}_1 x_2 x_3$ ✓

**Cover** : List of Prime Implicants that account for all $f = 1$.

**Cost** : Number of gates (excluding not gate on literals) and number of inputs to each gate.

**Example 2.** *Find minimum cost expression for the function* $f(x_1, x_2, x_3) = \prod M(4,5,6)$

(1) Find SOP




(1) Find min SOP

(2) " min POS

(3) Choose best among them

PIs: $\bar{x}_1$, $x_2x_3$ , EPIs: $\bar{x}_1$, $x_2x_3$

$f = \bar{x}_1 + x_2x_3$

| $x_1$ | $x_2$ | $x_3$ | f |
|---|---|---|---|
| 0 | 0 | 0 | 0 |
| 0 | 0 | 1 | 1 |
| 0 | 1 | 0 | 0 |

**Problem 2.** *Find minimum cost expression for the function* $f(x_1, x_2, x_3) = \prod M(2,5,6)$

(1) Find SOP — Making a K-map

(2) Find POS — Finding EPIs + PIs → shortest

$f = \pi M(0,2,\ldots)$

### 4.1 Incompletely specified functions or Don't cares

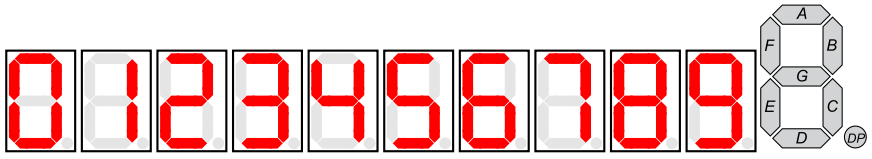
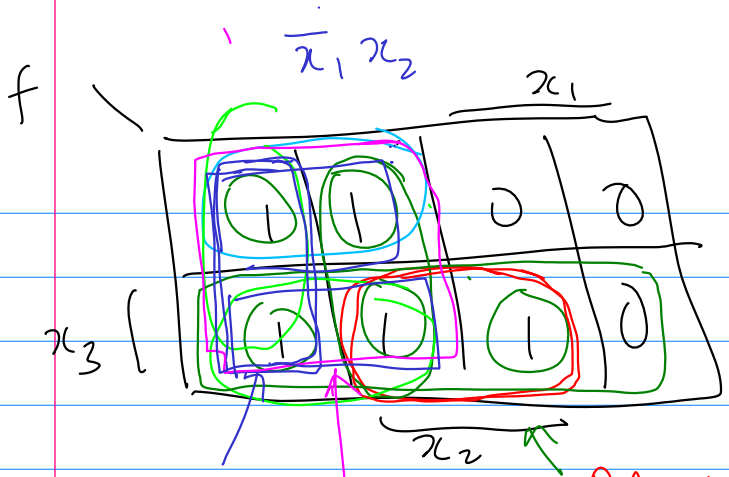
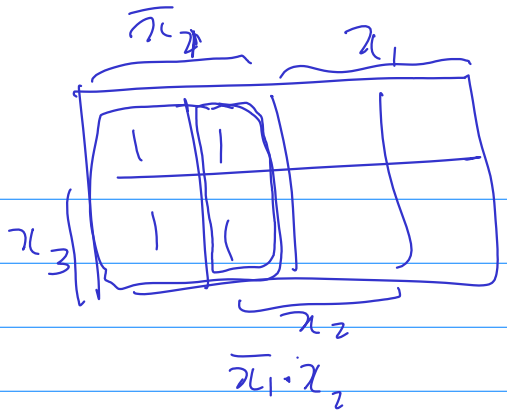



Figure 1: 7 Segment Representations of Each Integer



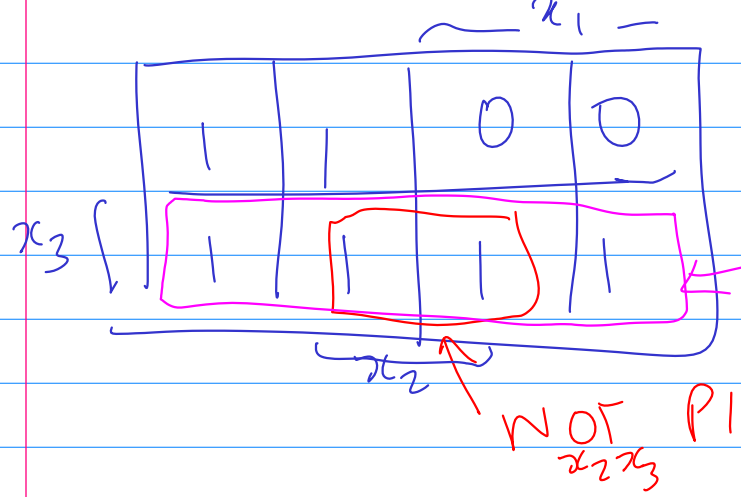




NOT PI
$= \bar{x}_1 \bar{x}_2$

PI $= \bar{x}_1$

PRIME IMPLICANT (fewest literals)
$= x_2 x_3$

= fewest inputs
= lowest cost



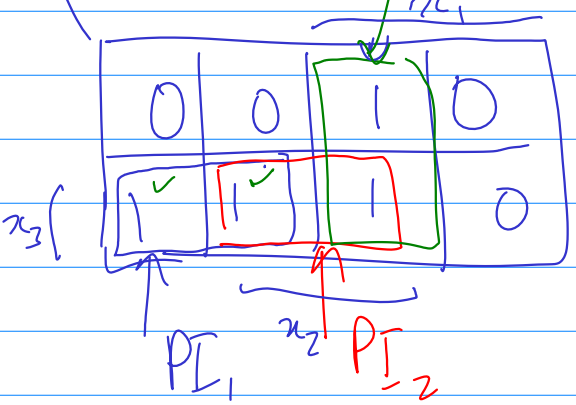

$PI = x_3$

NOT PI
$x_2 x_3$

EPI

lowers = 2, 4, 8, 16, 32 of 2

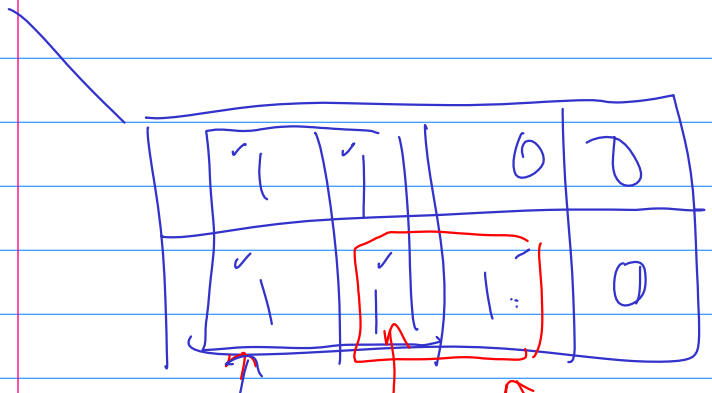
$PI_3$




$PI_1$

$PI_2$

$EPI_1$



Essential PI
EPI

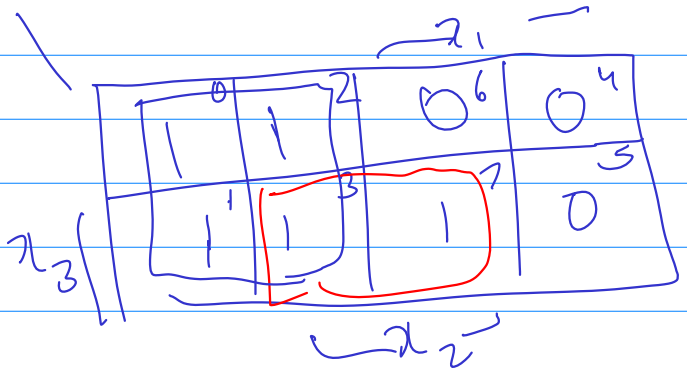
Essential PI
EPI

$$f(x_1, x_2, x_3) = \prod M(4, 5, 6)$$

Maxterms

| $x_1$ | $x_2$ | $x_3$ | f |
|---|---|---|---|
| 0 | 0 | 0 | 1 |
| 0 | 0 | 1 | 1 |
| 0 | 1 | 0 | 1 |
| 0 | 1 | 1 | 1 |
| 1 | 0 | 0 | 0 |
| 1 | 0 | 1 | 0 |
| 1 | 1 | 0 | 0 |
| 1 | 1 | 1 | 1 |

f




① PIs
$\bar{x}_1, x_2 x_3$

② EPIs
$\bar{x}_1, x_2 x_3$

③ $f = \bar{x}_1 + x_2 x_3$

1 AND #2

1 OR gate #2

④ Stop

Cost of f = 2 gates + 4 inputs

= 6

| BCD Value | | | | LED Segment |
|---|---|---|---|---|
| $D_3$ | $D_2$ | $D_1$ | $D_0$ | E |
| 0 | 0 | 0 | 0 | 0 |
| 0 | 0 | 0 | 1 | 1 |
| 0 | 0 | 1 | 0 | 0 |
| 0 | 0 | 1 | 1 | 1 |
| 0 | 1 | 0 | 0 | 1 |
| 0 | 1 | 0 | 1 | 1 |
| 0 | 1 | 1 | 0 | 0 |
| 0 | 1 | 1 | 1 | 1 |
| 1 | 0 | 0 | 0 | 0 |
| 1 | 0 | 0 | 1 | 1 |
| 1 | 0 | 1 | 0 | d |
| 1 | 0 | 1 | 1 | d |
| 1 | 1 | 0 | 0 | d |
| 1 | 1 | 0 | 1 | d |
| 1 | 1 | 1 | 0 | d |
| 1 | 1 | 1 | 1 | d |

**Example 3.** *Find minimum cost expression for the function*

$$f(x_1, \ldots, x_4) = \sum m(2, 4, 5, 6, 10) + D(12, 13, 14, 15)$$

**Problem 3.** *Find minimum cost expression for the function*

$$f(x_1, \ldots, x_4) = \sum m(0, 2, 4, 6, 7, 8, 9, 13) + D(1, 12, 15)$$

SOP form

Min. cost expression from K-map

① Find all PI

② Find all EPI

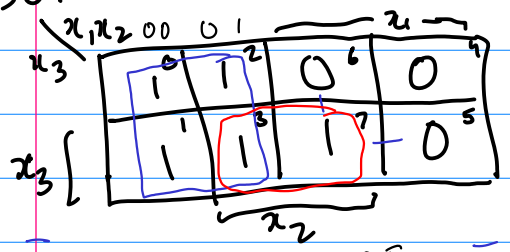
③ $f = \sum EPI$ +

④ Do EPIs ---- cover all $f=1$

if ④ is false

Petrick's methods

1 Find SOP




PIs: $\bar{x}_1$, $x_2x_3$ , EPIs: $\bar{x}_1$, $x_2x_3$

$f = \bar{x}_1 + x_2x_3$

→ (1) Find min SOP
(2) " min POS
(3) Choose best among them

| $x_1$ | $x_2$ | $x_3$ | $f$ |
|---|---|---|---|
| 0 | 0 | 0 | |

2 Find POS

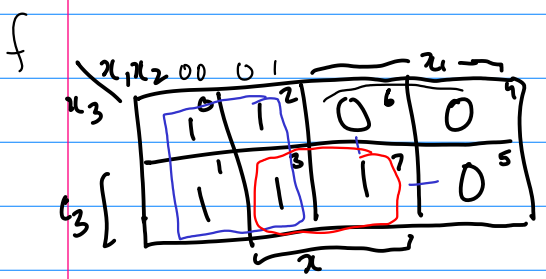
Demorgan theorem

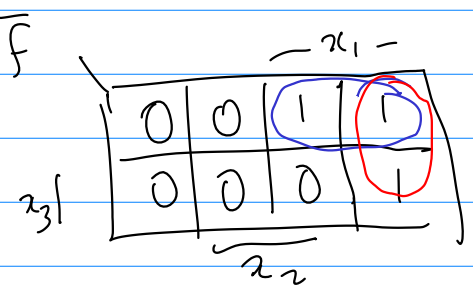
① Obs.: POS is a Demorgan complement of SOP

$f = \bar{x}_1 + x_2x_3$ ← SOP

Take complement on both sides

$\bar{f} = \overline{\bar{x}_1 + x_2x_3} = \bar{\bar{x}}_1 \cdot \overline{x_2x_3} = x_1 \cdot (\bar{x}_2 + \bar{x}_3)$ ← POS




$\bar{f}$ = SOP

↓ Demorgan theorem

$f$ = POS

PIs = $x_1\bar{x}_3$ , $x_1\bar{x}_2$

EPIs = $x_1\bar{x}_3$ , $x_1\bar{x}_2$

$f = x_1x_3 + x_1\bar{x}_2$

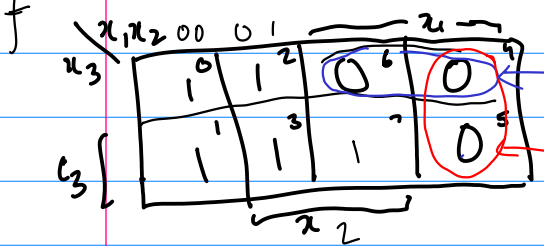
$\bar{\bar{f}} = \overline{x_1\bar{x}_3 + x_1\bar{x}_2}$

$f = (\bar{x}_1 + x_3) \cdot (\bar{x}_1 + x_2)$ ← POS

$f = (\bar{x}_1 + x_3)(x_2 + \bar{x}_1)$

configurations

Maxterms




Following rules for labeling maxterms

$f = \bar{x}_1 + x_2 x_3$ ← SOP

1 + 2 in

1 + 2 in

$f = (\bar{x}_1 + x_3) \cdot (\bar{x}_1 + x_2)$ ← POS

1g + 2in  1g + 2in

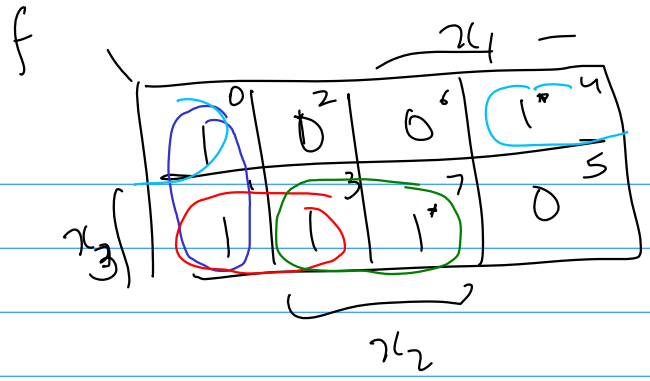
1g + 2in

Cost of SOP form = 3 + 3 = 6

Cost of POS form = 3 + 3 + 3 = 9

Min cost expression in SOP form $f = \bar{x}_1 + x_2 x_3$

$f = \Pi M(2,5,6)$




PIs: $\bar{x}_1\bar{x}_2$, $\bar{x}_1 x_3$, $x_2 x_3$, $\bar{x}_2\bar{x}_3$

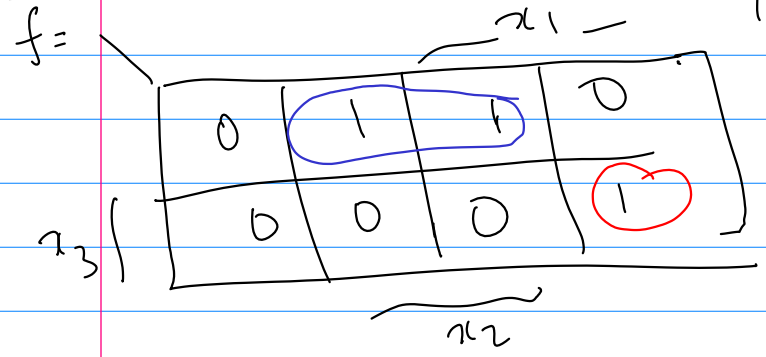
EPIs: $x_2 x_3$, $\bar{x}_2\bar{x}_3$

$f = x_2 x_3 + \bar{x}_2\bar{x}_3 + \bar{x}_1\bar{x}_2$ ← min SOP

min POS

EPIs = PIs = $x_2\bar{x}_3$, $x_1\bar{x}_2 x_3$

$\bar{f} =$




$\bar{f} = x_2\bar{x}_3 + x_1\bar{x}_2 x_3$

↓ Demorgan's complement

min POS $f = (\bar{x}_2 + x_3)\cdot(\bar{x}_1 + x_2 + \bar{x}_3)$

cost of SOP form: $\underbrace{x_2 x_3}_{1g+2in} + \underbrace{\bar{x}_2\bar{x}_3}_{1g+2in} + \underbrace{\bar{x}_1\bar{x}_2}_{1g+2in}$

$1g+3in$

$= 3+3+3+4 = 13$

Cost of POS form: $f = (\bar{x}_2 + x_3) \cdot (\bar{x}_1 + x_2 + \bar{x}_3)$

$1g + 2$ inputs $\quad\quad 1g + 3$ inputs

$1g + 2$ inputs

$= 3 + 4 + 3 = 10$

Min cost exp is POS form

$f = (\bar{x}_2 + x_3) \cdot (\bar{x}_1 + x_2 + \bar{x}_3)$