

POS

$$f(A, B, C, D) = (A + B)(C + D)$$

$$S(A, B, C, D) = (A + B + \bar{C} + \bar{D})(\bar{A} + \bar{B} + C + D)$$

NAND/NOR gates + Quine McCluskey + Petricks

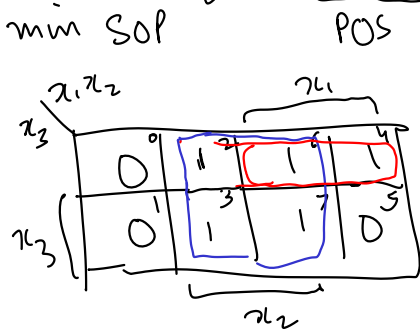
Maxterm

Vikas Dhiman for ECE275

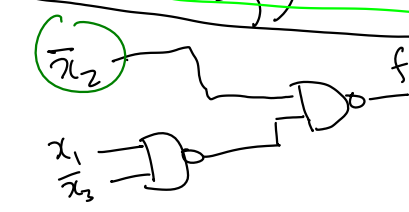
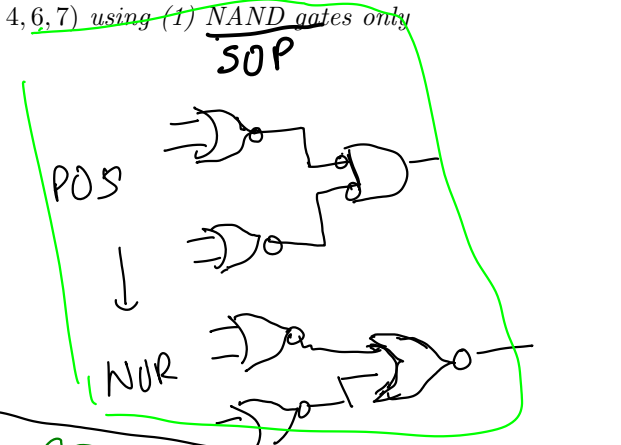
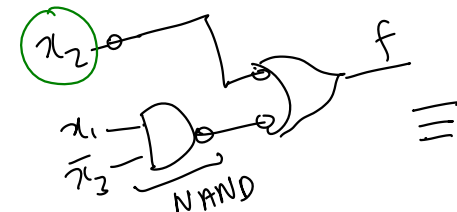
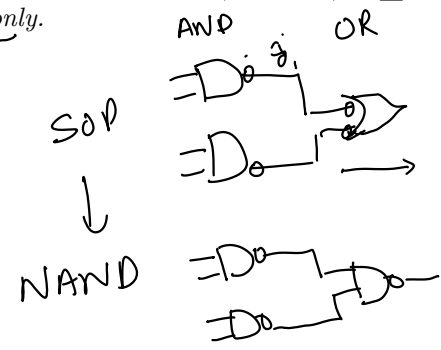
September 28, 2022

1 Circuit design using NAND/NOR gates

Example 1. Implement the function $f(x_1, x_2, x_3) = \sum m(2, 3, 4, 6, 7)$ using (1) NAND gates only and (2) NOR gates only.

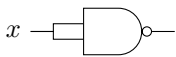


$$f = x_2 + x_1 x_3$$



Remark 1. NAND-NAND logic is generated from SOP form. NOR-NOR logic is generated from POS form.

Remark 2. NOT gate can also be created from a NAND gate $\bar{x} = \overline{x \cdot x}$.



Remark 3. NOT gate can also be created from a NOR gate $\bar{x} = \overline{x + x}$.



Problem 1. Design the simplest circuit that implements the function $f(x_1, x_2, x_3) = \sum m(3, 4, 6, 7)$ using (1) NAND gates only (2) NOR gates only.

2 Quine-McCluskey

This is not in the text-book. For additional reading, please refer to the linked resources on the website.

Definition 1 (Implicant). *Given a function f of n variables, a product term P is an implicant of f if and only if for every combination of values of the n variables for which $P = 1$, f is also equal to 1.*

Definition 2 (Prime Implicant). *A prime implicant of a function f is an implicant which is no longer an implicant if any literal is removed from it.*

There are 4 main steps in the Quine-McCluskey algorithm:

- 1. Generate Prime Implicants
- 2. Construct Prime Implicant Table. PIs as columns, and minterms as rows (don't cares are excluded).
- 3. Reduce Prime Implicant Table by repeating following steps until they it cannot be reduced further
 - (a) Remove Essential Prime Implicants
 - (b) Row Dominance: Remove *dominating* rows. (i.e. unnecessary minterms)
 - (c) Column Dominance: Remove *dominated* columns. (i.e. remove unnecessary PIs)
- 4. Solve Prime Implicant Table by Petricks method

2.1 Generate Prime Implicants

Example 2. *Generate prime implicants of the function $F(A, B, C, D) = \sum m(0, 2, 5, 6, 7, 8, 10, 12, 13, 14, 15)$ using Quine-McCluskey method*

| | | | |
|---|---|---|---|
| | 0 | 4 | 8 |
| 1 | 1 | | |
| 2 | | | |
| 3 | 1 | | |

Steps:

1. Start with writing minterms in binary format (include don't cares as minterms).
2. Create potential groups of minterms that can be combined (merged). The only minterms that can be combined differ only by single 1. Create a new list of combined minterms as n-1 literal implicants.

3. Check off the minterms that could be combined. Unchecked minterms are prime implicants (PIs).
4. Repeat the grouping process with n-1 literal implicants.

Problem 2. Generate PIs for the function $F(A, B, C, D) = \sum m(0, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13)$.

2.2 Prime Implicants table and reduction

Example 3. Reduce the prime implicants $\{\bar{B}\bar{D}, C\bar{D}, BD, BC, A\bar{D}, AB\}$ using prime implicants table.

Example 4. Make a PI table from the K-map and reduce the PI table to get min-SOP form

| | | | | | |
|------|------|------|------|------|------|
| | | AB | | | |
| | | 00 | 01 | 11 | 10 |
| CD | 00 | 1 | 1 | 0 | 0 |
| | 01 | 0 | 1 | 1 | 0 |
| | 11 | 0 | 0 | 1 | 1 |
| | 10 | 0 | 0 | 0 | 0 |

Example 5. Make a PI table from the K-map and reduce the PI table to get min-SOP form

| | | | | | |
|------|----|------|----|----|----|
| | | AB | | | |
| | | 00 | 01 | 11 | 10 |
| CD | 00 | d | 0 | 0 | 0 |
| | 01 | 1 | 1 | d | d |
| | 11 | 1 | 1 | 0 | 0 |
| | 10 | 1 | d | 0 | 0 |

Example 6. Reduce the following PI table and find the minimal SOP form

| | $\bar{A}\bar{D}$ | $\bar{B}\bar{D}$ | $\bar{C}\bar{D}$ | $\bar{A}C$ | $\bar{B}C$ | $\bar{A}B$ | $B\bar{C}$ | $A\bar{B}$ | $A\bar{C}$ |
|----|------------------|------------------|------------------|------------|------------|------------|------------|------------|------------|
| 0 | X | X | X | | | | | | |
| 2 | X | X | | X | X | | | | |
| 3 | | | | X | X | | | | |
| 4 | X | | X | | | X | X | | |
| 5 | | | | | | X | X | | |
| 6 | X | | | X | | X | | | |
| 7 | | | | X | | X | | | |
| 8 | | X | X | | | | | X | X |
| 9 | | | | | | | | X | X |
| 10 | | X | | | X | | | X | |
| 11 | | | | | X | | | X | |
| 12 | | | X | | X | | X | | X |
| 13 | | | | | | | X | | X |

2.3 Petricks method

Example 7. Solve the Prime Implicant table using Petrick's method and find the min-SOP form

| | $p_1 = \bar{A}C$ | $p_2 = \bar{B}C$ | $p_3 = \bar{A}B$ | $p_4 = B\bar{C}$ | $p_5 = A\bar{B}$ | $p_6 = A\bar{C}$ |
|----|------------------|------------------|------------------|------------------|------------------|------------------|
| 3 | X | X | | | | |
| 5 | | | X | X | | |
| 7 | X | | X | | | |
| 9 | | | | | X | X |
| 11 | | X | | | X | |
| 13 | | | | X | | X |

Example 8. Find the minimum SOP expression for the function $F(A, B, C, D) = \sum m(2, 3, 7, 9, 11, 13) + \sum d(1, 10, 15)$ using Quine-McCluskey method.

min POS

- ✓ ① \bar{f}
- ✓ ② min SOP for \bar{f}
- ✓ ③ Take inverse on both sides
- ✓ ④ Apply DeMorgan's to get PO

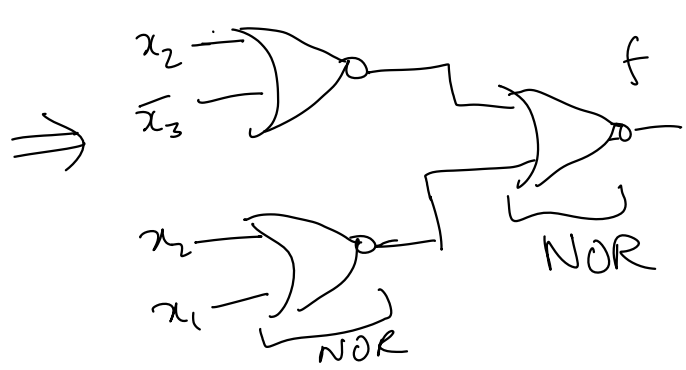
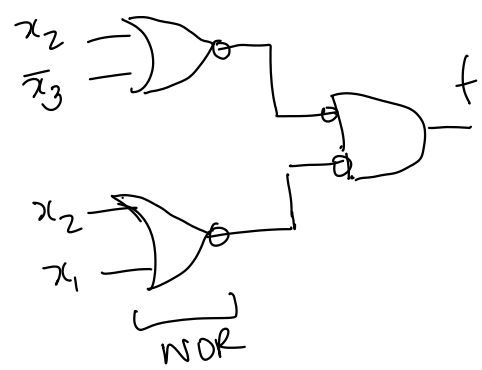
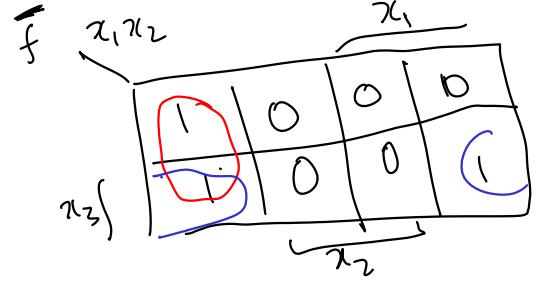
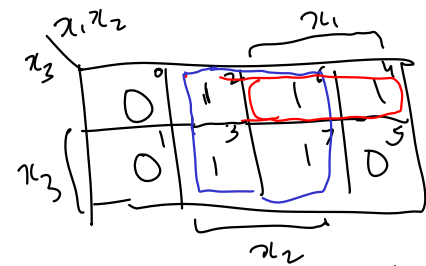
$$\bar{f} = \bar{x}_2 x_3 + \bar{x}_2 \bar{x}_1$$

$$\Rightarrow \bar{\bar{f}} = \overline{\bar{x}_2 x_3 + \bar{x}_2 \bar{x}_1}$$

$$\Rightarrow f = \overline{\bar{x}_2 x_3} \cdot \overline{\bar{x}_2 \bar{x}_1}$$

$$= (x_2 + \bar{x}_3) \cdot (x_2 + x_1) \leftarrow \text{min POS}$$

\bar{f} min SOP POS

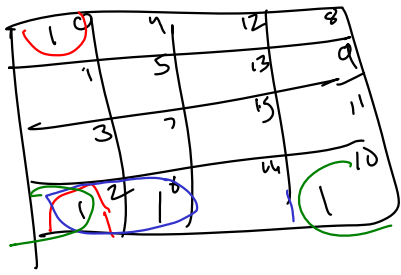


Quinn McEluskey

| Group # | Minterms | Representation |
|---------|----------|----------------|
| 0 | 0 | 0000 |
| 1 | 1 | 0001 |
| | 2 | 0010 |
| | 4 | 0100 |
| | 8 | 1000 |
| 2 | 3 | 0011 |
| | 5 | 0010 |
| | 6 | 1100 |
| | 9 | 1001 |
| | 10 | 0100 |
| | 12 | 1100 |
| 3 | 7 | 0111 |
| | 11 | 1011 |
| | 13 | 1101 |
| | 14 | 1110 |
| 4 | 15 | 1111 |

$$F(A, B, C, D) = \sum m(0, 2, 5, 6, 7, 8, 10, 12, 13, 14, 15)$$

| Group # | Minterms | Repr | 3-literal implicant minterms | Representation |
|---------|----------|--------|------------------------------|----------------|
| 0 | 0 | 0000 ✓ | (0,2) | 00-0 |
| 1 | 2 | 0010 ✓ | (0,8) | -000 |
| | 8 | 1000 ✓ | (2,6) | 0-10 |
| | 5 | 0101 ✓ | (2,10) | -010 |
| | 6 | 0110 ✓ | (8,10) | 10-0 |
| | 10 | 1010 ✓ | (8,12) | 1-00 |
| | 12 | 1100 ✓ | (5,7) | 01-1 |
| 2 | 7 | 0111 ✓ | (5,13) | -101 |
| | 13 | 1101 ✓ | (6,7) | 01-1 |
| | 14 | 1110 | (6,14) | -110 |
| 3 | 15 | 1111 | (10,14) | 1-10 |
| 4 | 15 | 1111 | (12,13) | 110- |
| | | | (12,14) | 11-0 |



| Group # | Minterms | Repr |
|---------|----------|--------|
| 0 | 0 | 0000 ✓ |
| 1 | 2 | 0010 ✓ |
| | 8 | 1000 ✓ |
| | 5 | 0101 ✓ |
| | 6 | 0110 ✓ |
| 2 | 10 | 1010 ✓ |
| | 12 | 1100 ✓ |
| | 7 | 0111 ✓ |
| 3 | 13 | 1101 ✓ |
| | 14 | 1110 ✓ |
| 4 | 15 | 1111 ✓ |

| 3-literal implicant minterms | Representation |
|------------------------------|----------------|
| (0,2) | 00-00 ✓ |
| (0,8) | -000 ✓ |
| (2,6) | 0-10 ✓ |
| (2,10) | -010 ✓ |
| (8,10) | 10-00 ✓ |
| (8,12) | 1-00 ✓ |
| (5,7) | 01-1 ✓ |
| (5,13) | -101 ✓ |
| (6,7) | 011- ✓ |
| (6,14) | -110 ✓ |
| (10,14) | 1-10 ✓ |
| (12,13) | 110- ✓ |
| (12,14) | 11-0 ✓ |
| (7,15) | -111 ✓ |
| (13,15) | 11-1 ✓ |
| (14,15) | 111- ✓ |

2-literal implicants

Minterms | Representation

$\bar{B} \bar{D}$

| | |
|-------------------------|-----------------|
| (0,2,8,10) | -0-0 |
| (0,8,2,10) | -0-0 |
| (2,6,10,14) | --10 |
| (2,10,6,14) | --10 |
| (8,10,12,14) | 1--0 |
| (8,12,10,14) | 1--0 |
| (12,13,14,15) | 11-- |

① Dash should match exactly
 ② exactly 1-bit should differ
 should differ by exactly 1-bit

| | |
|-----------------------------|------|
| (5, 7, 13, 15) | -1-1 |
| (5, 13, 7, 15) | -1-1 |
| (6, 7, 14, 15) | -11- |
| (6, 14, 7, 15) | -11- |
| (12, 14, 13, 15) | |

2-literal implicants

| Minterms | Repr |
|------------------|-------------------------|
| (0, 2, 8, 10) | -0-0 = $\bar{B}\bar{D}$ |
| (2, 6, 10, 14) | --10 = $C\bar{D}$ |
| (8, 10, 12, 14) | 1--0 = $A\bar{D}$ |
| (12, 13, 14, 15) | 11-- = AB |
| (5, 7, 13, 15) | -1-1 = BD |
| (6, 7, 14, 15) | -11- = BC |

1-literal implicant



PIs = { $\bar{B}\bar{D}$, $C\bar{D}$, $A\bar{D}$, AB , BD , BC }

Verilog { find minimum cost expression is NP-complete / NP-hard
 computation time $\propto \exp(n)$
 approximation

PI table

| minterms ↓ | PIs → | | | | | |
|---------------|--|------------------------|------------------------|------------------------|-------------------------|--------------------------|
| | $\bar{B}\bar{D}$ | $C\bar{D}$ | BD | BC | $A\bar{D}$ | AB |
| | -0-0 0000 = 0 0010 = 2 1000 = 8 1010 = 10 (0, 2, 8, 10) | --10 (2, 6, 10, 14) | -1-1 (5, 7, 13, 15) | -11- (6, 7, 14, 15) | 1--0 (8, 10, 12, 14) | 11-- (12, 13, 14, 15) |

PI table

| minterms | $\bar{B}\bar{D}$ | $C\bar{D}$ | $B\bar{D}$ | BC | $A\bar{D}$ | AB |
|----------|--|----------------------------|----------------------------|----------------------------|-----------------------------|-----------------------------|
| | $-0-0$ $0000 = 0$ $0010 = 2$ $1000 = 8$ $1010 = 10$ $(0, 2, 8, 10)$ | $--10$ $(2, 6, 10, 14)$ | $-1-1$ $(5, 7, 13, 15)$ | $-11-$ $(6, 7, 14, 15)$ | $1--0$ $(8, 10, 12, 14)$ | $11--$ $(2, 13, 14, 15)$ |
| 0* | X | | | | | |
| 2 | X | X | | | | |
| 5* | | | X | | | |
| 6 | | X | | X | | |
| 7 | | | X | X | | |
| 8 | X | | | | X | |
| 10 | X | X | | | X | |
| 12 | | | | | X | X |
| 13 | | | X | | | X |
| 14 | | X | | X | X | X |
| 15 | | | X | X | | X |

(1) Find EPIs = $\{\bar{B}\bar{D}, B\bar{D}\}$

Remove EPIs and covered minterms from the PI table

(2) Remove dominating Rows

(3) Remove dominated Columns.

| | $C\bar{D}$ | BC | $A\bar{D}$ | AB |
|--------|------------|------|------------|------|
| 6^P | x | x | | |
| 12^P | | | x | x |
| 14 | x | x | x | x |

(2) Row 14 dominates row 6
" " " " " 12
co-dominates

(3) Column $C\bar{D}$ dominates BC
" AB dominates $A\bar{D}$

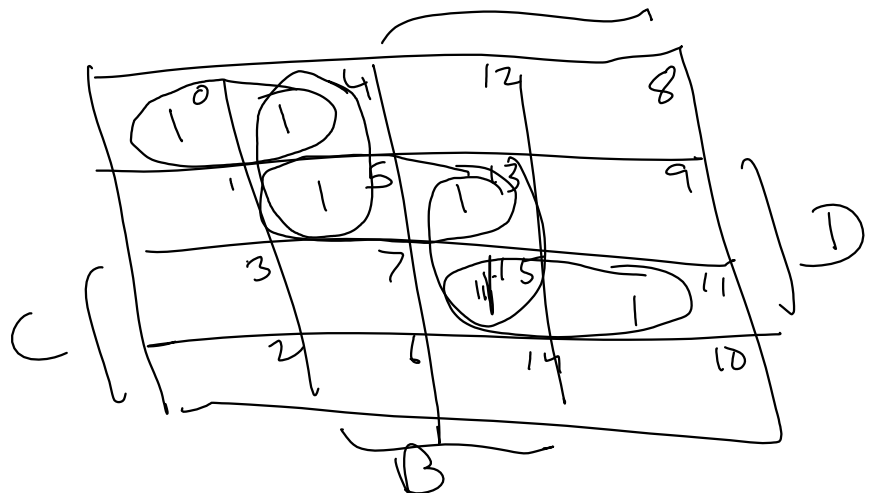
(A.1) Find secondary EPIs
 $= \{C\bar{D}, AB\}$

$$f = B\bar{D} + BD + C\bar{D} + AB$$

min SOP form

Example 4 PI table

| minterms | PIs \rightarrow | | | | |
|----------|-------------------------|-------------------|-------------|-------|-------|
| | $\bar{A}\bar{C}\bar{D}$ | $\bar{A}B\bar{C}$ | $B\bar{C}D$ | ABD | ACD |
| 0* | x | | | | |
| 4 | x | x | | | |
| 5 | | x | x | | |
| 11* | | | | | x |
| 13 | | | x | x | |
| 15 | | | | x | x |



① Find/Remove EPIs
 $= \{ \bar{A}\bar{C}\bar{D}, ACD \}$

②

| | PI | | |
|----|-------------------------|-------------|-------|
| | $\bar{A}\bar{B}\bar{C}$ | $B\bar{C}D$ | ABD |
| 5 | | x | |
| 13 | | x | x |

② Remove dominating rows

③ Remove dominated columns

$B\bar{C}D$ dominates $\bar{A}\bar{B}\bar{C}$

||

||

ABD

① Find secondary EPIs = $\{ B\bar{C}D \}$
 $f = \bar{A}\bar{C}\bar{D} + ACD + B\bar{C}D$ } min SOP

Example 5

Construct PI table and then reduce it

① Finding PI don't cares = minterms

② Finding cover exclude don't cares

| PI # minterms | PI \rightarrow $\bar{A}\bar{B}$ | $\bar{C}D$ | $\bar{A}D$ | $\bar{A}C$ |
|------------------|--------------------------------------|------------|------------|------------|
| ↓ | | | | |
| 1 | x | x | x | |
| 2 | x | | | x |
| 3 | x | | x | x |
| 5 | | x | x | |
| 7 | | | x | x |

| | | A | | | |
|---|----|----------------|----------------|----|----|
| | | 00 | 01 | 11 | 10 |
| C | 00 | d ⁰ | 0 ⁴ | 0 | 0 |
| | 01 | 1 ¹ | 1 ⁵ | d | d |
| | 11 | 1 ³ | 1 ⁷ | 0 | 0 |
| | 10 | 1 ² | d ⁸ | 0 | 0 |

B

① Find Essential PIs

② Find row dominance

Row ① dominates ⑤
 Row ③ dominates ② and ⑦

 Remove dominating Rows

③ Find column dominance

col $\bar{A}D$ dominates
 $\bar{C}D$
 $\bar{A}B$

 Remove dominated columns

| | $\bar{A}D$ | $\bar{A}C$ |
|----|------------|------------|
| 2* | | \times |
| 5* | \times | |
| 7 | \times | \times |

2.1 Find secondary EPI = $\{ \bar{A}C, \bar{A}D \}$

$$f = \bar{A}C + \bar{A}D$$

Example 7. Solve the Prime Implicant table using Petrick's method

| | $p_1 = \bar{A}C$ | $p_2 = \bar{B}C$ | $p_3 = \bar{A}B$ | $p_4 = B\bar{C}$ | $p_5 = A\bar{B}$ | $p_6 = A\bar{C}$ |
|----|------------------|------------------|------------------|------------------|------------------|------------------|
| 3 | X | X | | | | |
| 5 | | | X | X | | |
| 7 | X | | X | | | |
| 9 | | | | | X | X |
| 11 | | X | | | X | |
| 13 | | | | X | | X |

Find a cover

↳ Find minimum set of PI that covers all the minterms

Define a Boolean variable P_i

$P_i = \begin{cases} \text{True (1)} & \text{if } p_i \text{ is included} \\ \text{False (0)} & \text{otherwise} \end{cases}$ in the cover

$$f = \underbrace{\bar{A}C}_{p_1} + \underbrace{\bar{B}C}_{p_2} \quad p_3 = 0, p_4 = 0, p_5 = 0, p_6 = 0$$