POS

$f(A,B,C,D) = (A+B)(C+D)$

$g(A,B,C,D) = (A+B+\bar{C}+\bar{D})(\bar{A}+\bar{B}+C+D)$

Maxterm

NAND/NOR gates + Quine McCluskey + Petricks

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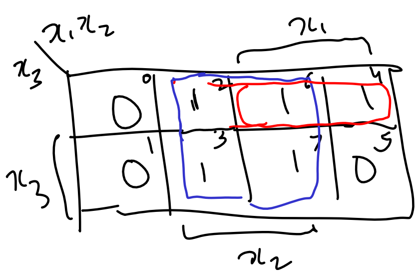
1 Circuit design using NAND/NOR gates

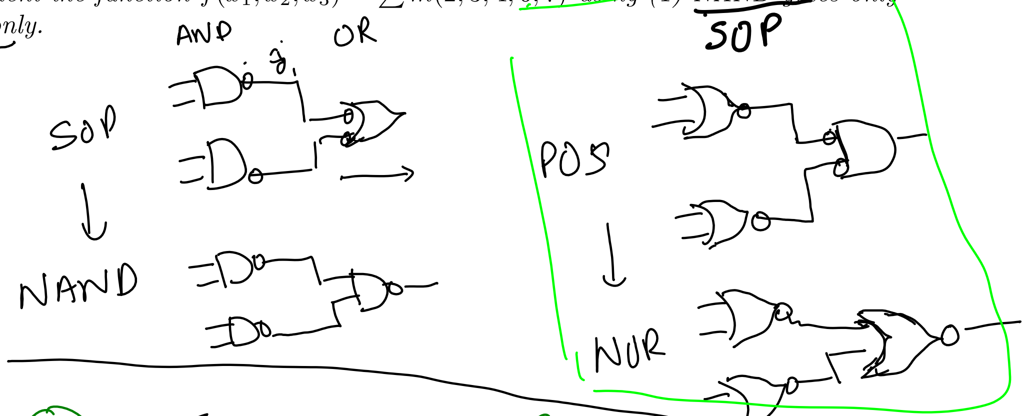
minterm

Example 1. *Implement the function $f(x_1, x_2, x_3) = \sum m(2, 3, 4, 6, 7)$ using (1) NAND gates only and (2) NOR gates only.*

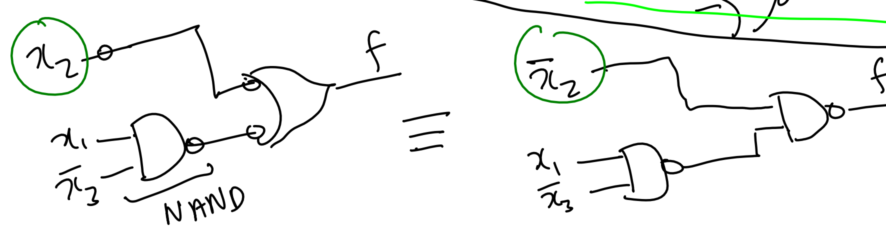
SOP

min SOP POS



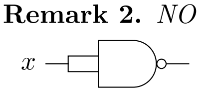


$f = x_2 + x_1\bar{x}_3$

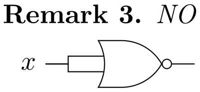


Remark 1. *NAND-NAND logic is generated from SOP form. NOR-NOR logic is generated from POS form.*

Remark 2. *NOT gate can also be created from a NAND gate $\bar{x} = \overline{x \cdot x}$.*



Remark 3. *NOT gate can also be created from a NOR gate $\bar{x} = \overline{x + x}$.*



Problem 1. *Design the simplest circuit that implements the function $f(x_1, x_2, x_3) = \sum m(3, 4, 6, 7)$ using (1) NAND gates only (2) NOR gates only.*

2 Quine-McCluskey

This is not in the text-book. For additional reading, please refer to the linked resources on the website.

Definition 1 (Implicant)**.** *Given a function f of n variables, a product term P is an implicant of f if and only if for every combination of values of the n variables for which $P = 1$, f is also equal to 1.*

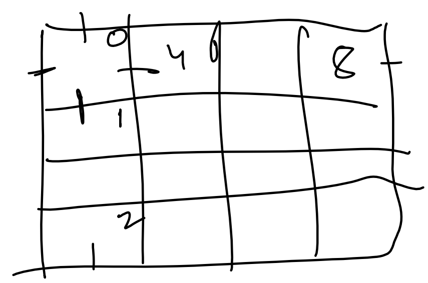
Definition 2 (Prime Implicant)**.** *A prime implicant of a function f is an implicant which is no longer an implicant if any literal is removed from it.*

There are 4 main steps in the Quine-McCluskey algorithm:

1. Generate Prime Implicants
2. Construct Prime Implicant Table. PIs as columns, and minterms as rows (don't cares are excluded).
3. Reduce Prime Implicant Table by repeating following steps until they it cannot be reduced further
   (a) Remove Essential Prime Implicants
   (b) Row Dominance: Remove *dominating* rows. (i.e. unnecessary minterms)
   (c) Column Dominance: Remove *dominated* columns. (i.e. remove unnecessary PIs)
4. Solve Prime Implicant Table by Petricks method

2.1 Generate Prime Implicants

Example 2. *Generate prime implicants of the function $F(A, B, C, D) = \sum m(0, 2, 5, 6, 7, 8, 10, 12, 13, 14, 15)$ using Quine-McCluskey method*



Steps:

1. Start with writing minterms in binary format (include don't cares as minterms).
2. Create potential groups of minterms that can be combined (merged). The only minterms that can be combined differ only be single 1. Create a new list of combined minterms as n-1 literal implicants.

3. Check off the minterms that could be combined. Unchecked minterms are prime implicants (PIs).

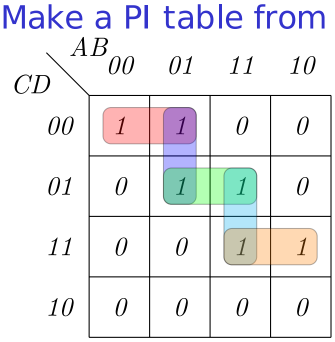
4. Repeat the grouping process with n-1 literal implicants.

Problem 2. *Generate PIs for the function* $F(A, B, C, D) = \sum m(0, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13)$.

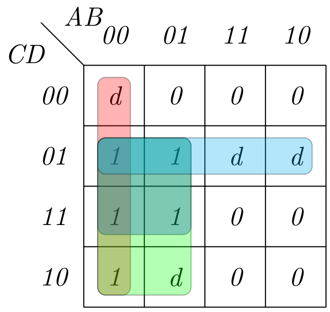
2.2 Prime Implicants table and reduction

Example 3. *Reduce the prime implicants* $\{\bar{B}\bar{D}, C\bar{D}, BD, BC, A\bar{D}, AB\}$ *using prime implicants table.*

Example 4. Make a PI table from the K-map and reduce the PI table to get min-SOP form



Example 5. Make a PI table from the K-map and reduce the PI table to get min-SOP form



Example 6. *Reduce the following PI table* and find the minimal SOP form

	$\bar{A}\bar{D}$	$\bar{B}\bar{D}$	$\bar{C}\bar{D}$	$\bar{A}C$	$\bar{B}C$	$\bar{A}B$	$B\bar{C}$	$A\bar{B}$	$A\bar{C}$
0	X	X	X						
2	X	X		X	X				
3				X	X				
4	X		X			X	X		
5						X	X		
6	X			X		X			
7				X		X			
8		X	X					X	X
9								X	X
10		X			X			X	
11					X			X	
12			X		X		X		X
13							X		X

2.3 Petricks method

Example 7. *Solve the Prime Implicant table using Petrick's method* and find the min-SOP form

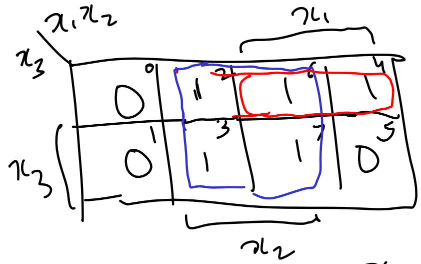
	$p_1 = \bar{A}C$	$p_2 = \bar{B}C$	$p_3 = \bar{A}B$	$p_4 = B\bar{C}$	$p_5 = A\bar{B}$	$p_6 = A\bar{C}$
3	X	X				
5			X	X		
7	X		X			
9					X	X
11		X			X	
13				X		X

Example 8. *Find the minimum SOP expression for the function* $F(A,B,C,D) = \sum m(2,3,7,9,11,13) + \sum d(1,10,15)$ *using Quine-McCluskey method.*

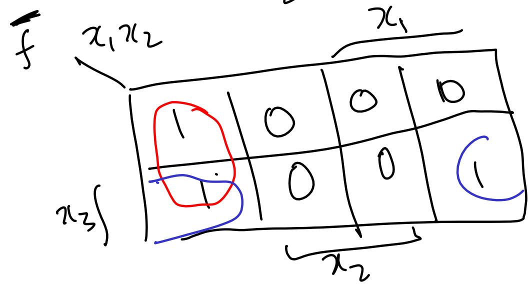
min POS

min SOP POS

f



$\bar{f}$



✓ (1) $\bar{f}$

✓ (2) min SOP for $\bar{f}$

✓ (3) Take inverse on both sides

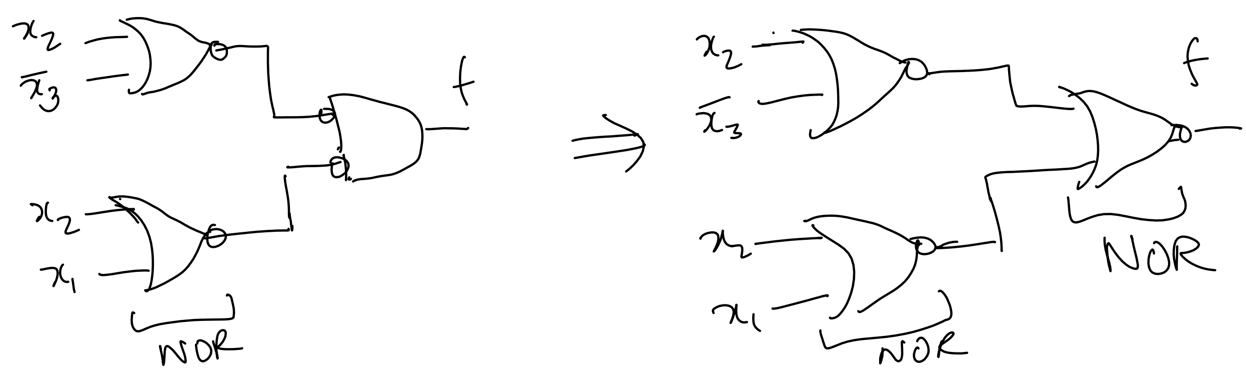
✓ (4) Apply Demorgan's to get PO

$$\bar{f} = \bar{x}_2 x_3 + \bar{x}_2 \bar{x}_1$$

$$\Rightarrow \bar{\bar{f}} = \overline{\bar{x}_2 x_3 + \bar{x}_2 \bar{x}_1}$$

$$\Rightarrow f = \overline{\bar{x}_2 x_3} \cdot \overline{\bar{x}_2 \bar{x}_1}$$

$$= (x_2 + \bar{x}_3) \cdot (x_2 + x_1) \leftarrow \text{min POS}$$



Quine McCluskey

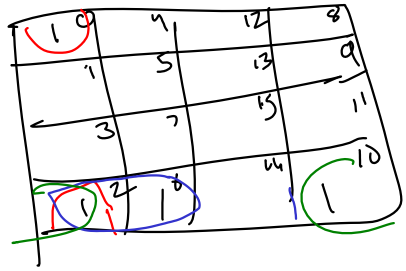
Group #	Minterms	Representation
0	0	0000
1	1 2 4 8	0001 0010 0100 1000
2	3 5 6 9 10 12	0011 0101 0110 1001 1010 1100
3	7 11 13 14	0111 1011 1101 1110
4	15	1111

$F(A, B, C, D) = \sum m(0, 2, 5, 6, 7, 8, 10, 12, 13, 14, 15)$ ✓

Group #	Minterms	Repr
0	0	0000 ✓
1	2 8	0010 ✓ 1000 ✓
2	5 6 10 12	0101 ✓ 0110 ✓ 1010 ✓ 1100 ✓
3	7 13 14	0111 ✓ 1101 ✓ 1110
4	15	1111

3-literal implicant

Minterms	Representation
(0,2)	00-0
(0,8)	-000
(2,6)	0-10
(2,10)	-010
(8,10)	10-0
(8,12)	1-00
(5,7)	01-1
(5,13)	-101
(6,7)	01-1
(6,14)	-110
(10,14)	1-10
(12,13)	110-
(12,14)	11-0



Group #	Minterms	Repr
0	0	0000 ✓
1	2	0010 ✓
	8	1000 ✓
2	5	0101 ✓
	6	0110 ✓
	10	1010 ✓
	12	1100 ✓
3	7	0111 ✓
	13	1101 ✓
	14	1110 ✓
4	15	1111 ✓

$\bar{A}\bar{B}\bar{D}$

3-literal implicant

Minterms	Represent
? (0,2)	00-0 ✓
(0,8)	-000 ✓
(2,6)	0-10 ✓
(2,10)	-010 ✓
? (8,10)	10-0 ✓
(8,12)	1-00 ✓
(5,7)	01-1 ✓
(5,13)	-101 ✓
(6,7)	011- ✓
(6,14)	-110 ✓
(10,14)	1-10 ✓
(12,13)	110- ✓
12,14	11-0 ✓
(7,15)	-111 ✓
(13,15)	11-1 ✓
(14,15)	111- ✓

2-literal implicants

$\bar{B}\bar{D}$

Minterms	Representation
(0,2,8,10)	-0-0
~~(0,8,2,10)~~	~~-0-0~~
(2,6,10,14)	--10
~~2,10,6,14~~	~~--10~~
(8,10,12,14)	1--0
~~(8,12,10,14)~~	~~1--0~~
(12,13,14,15)	11--

① Dash should match exactly
② exactly 1-bit should differ

should differ on the implicant by exactly 1-bit

(5, 7, 13, 15)	-1-1
~~(5, 13, 7, 15)~~	-1-1
(6, 7, 14, 15)	-11-
~~(6, 14, 7, 15)~~	~~-11-~~
~~(12, 14, 13, 15)~~	

2-literal implicants

Minterms	Repr
(0, 2, 8, 10)	$-0-0 = \bar{B}\bar{D}$
(2, 6, 10, 14)	$--10 = C\bar{D}$
(8, 10, 12, 14)	$1--0 = A\bar{D}$
(12, 13, 14, 15)	$11-- = AB$
(5, 7, 13, 15)	$-1-1 = BD$
(6, 7, 14, 15)	$-11- = BC$

1-literal implicant

$\times$

$PIs = \{ \bar{B}\bar{D}, C\bar{D}, A\bar{D}, AB, BD, BC \}$

Verilog: find minimum cost expression is NP-complete / NP-hard

computation time $\propto \exp(n)$

→ approximation

PI table

minterms ↓ PIs →

$\bar{B}\bar{D}$	$C\bar{D}$	BD	BC	$A\bar{D}$	AB
-0-0 0000 = 0 0010 = 2 1000 = 8 1010 = 10 (0, 2, 8, 10)	--10 (2, 6, 10, 14)	-1-1 (5, 7, 13, 15)	-11- (6, 7, 14, 15)	1--0 (8, 10, 12, 14)	11-- (12, 13, 14, 15)

PI table

PIs →

minterms ↓

minterms	$\bar{B}\bar{D}$	$C\bar{D}$	BD	BC	$A\bar{D}$	AB
	-0-0 0000 = 0 0010 = 2 1000 = 8 1010 = 10 (0,2,8,10)	--10 (2,6,10,14)	-1-1 (5,7,13,15)	-11- (6,7,14,15)	1--0 (8,10,12,14)	11-- (12,13,14,15)
0*	X					
2	X	X				
5*			X			
6		X		X		
7			X	X		
8	X				X	
10	X	X			X	
12					X	X
13			X			X
14		X		X	X	X
15			X	X		X

(1) Find EPIs = { $\bar{B}\bar{D}$, BD }

Remove EPIs and covered minterms from the PI table

② Remove dominating Rows

③ Remove dominated Columns.

	$C\bar{D}$	BC	$A\bar{D}$	AB
6	×	×		
12			×	×
14	×	×	×	×

② Row 14 dominates row 6

" " " " 12

co-dominate

③ Column $C\bar{D}$ dominates BC

" AB dominates $A\bar{D}$

(4.1) Find secondary EPIs

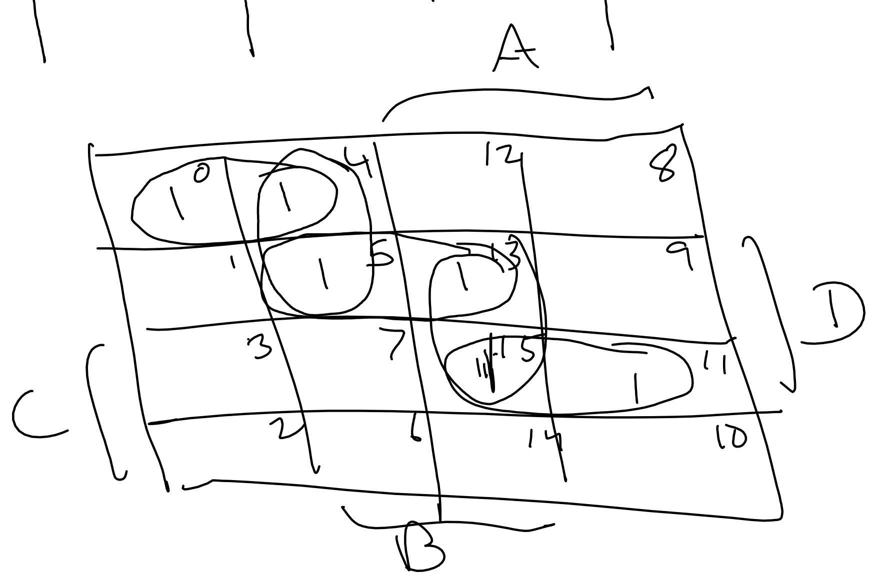
$= \{C\bar{D}, AB\}$

$$f = \bar{B}\bar{D} + BD + C\bar{D} + AB$$

min SOP form

Example 4 PI table

minterms ↓ \ PIs →	$\bar{A}\bar{C}\bar{D}$	$\bar{A}B\bar{C}$	$B\bar{C}D$	ABD	ACD
0*	×				
4	×	×			
5		×	×		
11*					×
13			×	×	
15				×	×



(1) Find/remove EPIs

$= \{\bar{A}\bar{C}\bar{D},\ ACD\}$

(2) PI

	$\bar{A}B\bar{C}$	$B\bar{C}D$	ABD
5	x	x	
13		x	x

(2) Remove dominating rows

(3) Remove dominated columns

$B\bar{C}D$ dominates $\bar{A}B\bar{C}$

" " ABD

(1) Find secondary EPIs $= \{B\bar{C}D\}$

$f = \bar{A}\bar{C}\bar{D} + ACD + B\bar{C}D$ } min SOP

Example 5

Construct PI table and then reduce it

① Finding PI don't cares = minterms

② Finding Cover exclude don't cares

PI ⟶

minterms ↓	$\bar{A}\bar{B}$	$\bar{C}D$	$\bar{A}D$	$\bar{A}C$
1	×	×	×	
2	×			×
3	×		×	×
5		×	×	
7			×	×

CD \ AB	00	01	11	10
00	d	0	0	0
01	1	1	d	d
11	1	1	0	0
10	1	d	0	0

① Find Essential PIs

② Find row dominance

Row ① dominates ⑤

Row ③ dominates ② and ⑦

Remove dominating Rows

③ Find column dominance

col $(\bar{A}D)$ dominates $\bar{C}D$

col $\bar{A}C$ dominates $\bar{A}\bar{B}$

Remove dominated columns

	$\bar{A}D$	$\bar{A}C$
2*		X
5*	X	
7	X	X

2.1 Find secondary EPI = $\{\bar{A}C, \bar{A}D\}$

$$f = \bar{A}C + \bar{A}D$$

Example 7. *Solve the Prime Implicant table using Petrick's method*

	$p_1 = \bar{A}C$	$p_2 = \bar{B}C$	$p_3 = \bar{A}B$	$p_4 = B\bar{C}$	$p_5 = A\bar{B}$	$p_6 = A\bar{C}$
	P_1	P_2	P_3			
3	X	X				
5			X	X		
7	X		X			
9					X	X
11		X			X	
13				X		X

$f = \underbrace{\bar{A}C}_{p_1} + \underbrace{\bar{B}C}_{p_2}$ $P_1 = 1$, $P_2 = 1$, $P_3 = 0$, $P_4 = 0$

Find a cover

↳ Find minimum set of PI that cover all the minterms

Define a Boolean variable P_i

$P_i = \begin{cases} \text{True}(1) \text{ if } p_i \text{ is included in the cover} \\ \text{False}(0) \text{ otherwise} \end{cases}$

$P_1 = 1$ $P_2 = 1$ $P_3 = 0$, $P_4 = 0$

$f = \underbrace{\bar{A}C}_{p_1} + \underbrace{\bar{B}C}_{p_2}$ $P_5 = 00$

Satisfy a boolean condition

= All minterms must be covered

= (m_3 must be covered)

AND (m_5 must be ")

AND (m_7 ")

⋮

AND (m_{13} ")

---

m_3 is covered = (P_1 is included)

OR (P_2 is included)

= $P_1 + P_2$

m_5 is covered $= P_3 + P_4$

All minterms are covered

$$= \underbrace{(P_1+P_2)}_{m_3 \text{ cover}} \underbrace{(P_3+P_4)}_{m_5 \text{ cover}} \underbrace{(P_1+P_3)}_{m_7 \text{ cover}} \overbrace{(P_5+P_6)}^{m_9 \text{ cover}}$$

$$\underbrace{(P_2+P_5)}_{m_{11} \text{ cover}} (P_4+P_6)$$

$$= (P_1P_3 + P_1P_4 + P_2P_3 + P_2P_4)$$
$$(P_1+P_3)(P_5+P_6)(P_2+P_5)(P_4+P_6)$$

$$= (P_1P_3 + P_1P_4 + P_1P_2P_3 + P_1P_2P_4$$
$$+ P_1P_3 + P_1P_3P_4 + P_2P_3 + P_2P_3P_4)$$
$$(P_4+P_6)(P_2+P_5)(P_5+P_6)$$

/

$$= \Big(P_1P_3P_4 + P_1P_4 + P_1P_2P_3P_4 + P_1P_2P_4 + P_1P_3P_4 + P_2P_3P_4 + P_2P_3P_4 + P_1P_3P_6 + P_1P_4P_6 + P_1P_2P_3P_6 + P_1P_2P_4P_6 + P_1P_3P_6 + P_1P_3P_4P_6 + P_2P_3P_6 + P_2P_3P_4P_6 \Big)$$

$$(P_2 + P_5)(P_5 + P_6)$$

$$= \Big(P_1P_3P_4 + P_1P_4 + P_1P_2P_3P_4 + P_1P_3P_4 + P_2P_3P_4 + P_1P_3P_6 + P_1P_2P_3P_6 + P_1P_3P_4P_6 + P_2P_3P_6 + P_2P_3P_4P_6 \Big)$$

$$(P_2 + P_3)(P_5 + P_6)$$

$$
\begin{aligned}
= \Big(& \underbrace{P_1P_2P_3}P_4 + P_1P_2P_4 + \cancel{\underbrace{P_1P_2P_3}P_4} + \cancel{\underbrace{P_1P_2P_3}P_4} \\
& + P_2P_3P_4 + P_1P_2P_3P_6 + \cancel{P_1P_2P_3P_6} \\
& + P_1P_2P_3P_4P_6 + P_2P_3P_6 + P_2P_3P_4P_6 \\
& + P_1P_3P_4P_5 + P_1P_4P_5 + P_1P_2P_3P_4P_5 \\
& + P_1P_3P_4P_5 + P_2P_3P_4P_5 + P_1P_3P_5P_6 \\
& + P_1P_2P_3P_5P_6 + P_1P_3P_4P_5P_6 \\
& + P_2P_3P_5P_6 + P_2P_3P_4P_5P_6 \Big) \left(P_5 + P_6\right)
\end{aligned}
$$

$=$

. . . . . — — - - . .

$$
= \overbrace{P_1P_2P_3}^{\text{feasible}} + \overbrace{P_2P_4P_5}^{\text{solution}} + P_5P_6P_7P_8
$$

$$
\begin{array}{cccc}
 & P_1 & P_2 & P_3
\end{array}
$$

$$f = P_1 + P_2 + P_3$$

$$= \bar{A}C + \bar{B}C + \bar{A}B$$

---

1. Find PIs using Quine McCluskey column procedure

2. Draw PI table

3. Reduce PI table
   - Find EPIs ; Remove corres. rows/cols
   - Remove dominating rows
   - Remove dominated cols

4. Find minimum cover using Petrick's method