Number system and conversions (section 1.4 of textbook)

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1 Place value number system

A decimal number 1342 can be written as

$$\begin{aligned} 1342 &= 1 \times 1000 + 3 \times 100 + 4 \times 10 + 2 \times 1 \\ &= 1 \times 10^3 + 3 \times 10^2 + 4 \times 10^1 + 2 \times 10^0. \end{aligned}$$

Decimal numbers are said to be of base (or radix) 10. One can generalize this idea to arbitrary base (or radix) r. The same number expressed in some other base r will have a very different value:

$$(1342)_r = 1 \times r^3 + 3 \times r^2 + 4 \times r^1 + 2 \times r^0.$$

In general the *value* of a arbitrary n-digit number $d_{n-1}d_{n-2}\dots d_1 d_0$ in base r is:

$$(d_{n-1}d_{n-2}\dots d_1 d_0)_r = d_{n-1} \times r^{n-1} + d_{n-2} \times r^{n-2} + \dots + d_1 \times r^1 + d_0 \times r^0 \qquad = \sum_{k=0}^{n-1} d_k r^k$$

Each digit value is always smaller than base $d_k <= r - 1$.

2 Binary numbers

Numbers with base 2 are called binary numbers. For example, the number $(10110)_2$ has value:

$$\begin{aligned} (10110)_2 &= 1 \times 2^4 + 0 \times 2^3 + 1 \times 2^2 + 1 \times 2^1 + 0 \times 2^0 \\ &= 22. \end{aligned}$$

Problem 1 *Convert the following binary numbers to decimal:* $(11110)_2$, $(100111)_2$.

$(11110)_2 = 1 \times 2^4 + 1 \times 2^3 + 1 \times 2^2 + 1 \times 2^1 + 0$

$= 30_{10}$

$(100111)_2 = 1 \times 2^5 + 0 + 0 + 1 \times 2^2 + 1 \times 2^1 + 1$

$= 39_{10}$

Conversion from decimal to binary The value is in decimal because we find it easy to do calculations in decimal numbers. Decimal values can be converted back to Binary representation by *repeated division* by 2 while noting down the remainder. Allow me to use / sign to denote both quotient and remainder after division. Let's convert $(22)_{10}$ back to binary:

$$
\begin{aligned}
22/2 &= (11, 0) && \text{11 is the quotient and 0 is the remainder} \\
11/2 &= (5, 1) && \text{5 is the quotient and 1 is the remainder} \\
5/2 &= (2, 1) \\
2/2 &= (1, 0) \\
1/2 &= (0, 1)
\end{aligned}
$$

Read the remainders from bottom to top and write them as left to right, to form the resultant binary number $(22)_{10} = (10110)_2$.

Problem 2 *Find the binary representation for decimal numbers: 123 and 89. Show your work.*

$123_{10} = (1111011)_2$

$89_{10} = (1011001)_2$

3 Hexadecimal numbers

Numbers with base 16 are called Hexadecimal numbers. From 0 to 9 the symbols are same as decimal numbers. From 10 to 15, Hexadecimal numbers use A to F.

$$A = 10, B = 11, C = 12, D = 13, E = 14, F = 15$$

. Example, $(10AD)_{16} = 1 \times 16^3 + 10 \times 16^1 + 13 = 4096 + 160 + 13 = 4269$.

4 Octal numbers

Numbers with base 8 are called octal numbers. Example, $(354)_8 = 3 \times 8^2 + 5 \times 8 + 4 = 192 + 40 + 4 = 236$.

5 Hexadecimal/octal to binary and vice-versa

Normally, if you have to convert between a number of base r_1 to a number of base r_2, we will have to convert it via decimal numbers. Convert from base r_1 to decimal and then from decimal to r_2.

Since Hexadecimal base 16 is an exact power of 2 ($16 = 2^4$). Conversion between Hexadecimal to binary is easy. You can group 4 binary digits from right to left and convert each group of 4 binary digits to a single Hexadecimal digit and back. Example, $(10110)_2 = (0001\_0110)_2 = (16)_{16}$. To convert back. Take example, $(10AD)_{16} = (0001\_0000\_1010\_1101)_2 = (1\_0000\_1010\_1101)_2$.

$16 = 2^4$

Problem 3 *Find the binary and decimal values of the following Hexadecimal numbers* $(A25F)_{16}$, $(F0F0)_{16}$.

$F = 1111$

$5 = 0101$

$2 = 0010$

$A = 1010$

$16^3 = 4096$

$16^2 = 256$

$(1010\ 0010\ 0101\ 1111)_2$

20831

41567

$F0F0 = 61680$

$16 = 2^4$

Similarly octal to binary can proceed by grouping 3-binary digits at a time. Example, $(354)_8 = (011\_101\_100)_2$.

$8 = 2^3$

Problem 4 *Find the binary and decimal values of the following Octal numbers* $(3751)_8$ *and* $(722)_8$.

$(3751)_8 = (011\ 111\ 101\ 001)_2$

$\le 8^2$ $0 - 7$

≤ 7

Binary $\xrightarrow{?}$ octal

$= 3 \times 8^3 + 7 \times 8^2 + \underbrace{5 \times 8^1} + \underbrace{1 \times 8^0}$

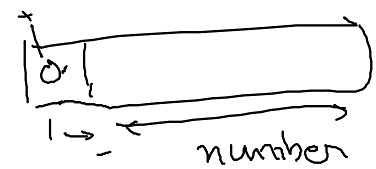
001

101 000

6 Signed binary numbers

Signed numbers include both negative and positive numbers. There three common signed number representations

1. Sign magnitude representation
2. One's complement
3. Two's complement



6.1 Sign-magnitude representation

The Most significant (left most) *bit* (binary digit) represents sign (0 = + and 1 = −), the rest represent the magnitude. Example, a 5-bit number $(11010)_2$ in signed magnitude representation has the value of $(-1010)_2 = -10$. Note that +10 has to be represented by a leading 0 at the most significant bit (MSB) $+10 = (01010)_2$. Hence, the number of bits have to be specified.

Problem 5 • *Write down all possible 4-digit binary numbers and corresponding decimal values if they are in signed magnitude format? What is the minimum and maximum value?*

- *What is the minimum and maximum value of n-digit signed binary number in sign-magnitude format?*

4-digit Binary		Decimal
0000	=	+0
0001	=	+1
0010	=	+2
0011	=	+3
0100	=	+4
0101	=	+5
0110	=	+6
0111	=	+7 ← Max $2^3 - 1$
1000	=	−0
1001	=	−1
1010	=	−2
1011	=	−3
1100	=	−4
1101	=	−5
1110	=	−6
1111	=	−7 ← Min $-(2^3 - 1)$

$(0000)_2 = (1000)_2$

n-digit

$2^{n-1} - 1$

$-(2^{n-1} - 1)$

$$(\underbrace{1101}\ \underbrace{1100})_2 \longrightarrow (\qquad)_{8=2^3}$$

$$(\underbrace{011}_{3}\ \underbrace{011}_{3}\ \underbrace{100}_{4})_2 \longrightarrow (3\ 3\ 4)_8$$

Number system to the base 3

0, 1, 2

$$(13)_{10} = (111)_3$$

3	13	
3	4 ,	1
3	1 ,	1
	0 ,	1

$9 + 4$

$3^2 + 3^1 + 1$

$1\cdot 3^2 + 1\cdot 3^1 + 1$

$$(13)_{10} = (\qquad)_8$$

$$(.111)_{64 = 8^2} \longrightarrow (\qquad)_8$$

$$(\qquad)_{10}$$

6.2 One's complement representation

In one's complement representation, the negative number is obtained by flipping all the bits of the corresponding positive number. Example, a 5-bit one's complement of $+10 = (01010)_2$ is $(10101)_2 = -10$. Note that flipping bits is equivalent to subtracting the number from $(11111)_2$, hence the name.

$+10 = (01010)_2$

1's comp

Problem 6 • *Write down all possible 4-digit binary numbers and corresponding decimal values if they are in sign magnitude format? What is the minimum and maximum value?*

- *What is the minimum and maximum value of n-digit signed binary number in one's complement?*

0000 = +0
0001 = +1
0010 = 2
0011
0100 = 4
0101
0110
0111 = +7
1000 = -7
1001 = -6
1010 = -5
1011 = -4
1100 = -3

1101 = -2
1110 = -1
1111 = -0

$\boxed{\begin{array}{l} 4 + (-5) = -1 \\ 0100 + 1010 \\ = 1110 = -1 \end{array}}$

$(1010) = -(0101)_2 = -5$

1's comp

6.3 Two's complement representation

In two's complement representation, the n-digit negative number is obtained by subtracting the positive number from 2^n. Example, two's complement of 5-digit binary number $+10 = (01010)_2$ is $2^5 - 10 = 22 = (11000)_2$. An easier algorithm to get two's complement goes via one's complement. Note that $(11111)_2 = 2^5 - 1$. We can get two's complement by adding 1 to one's complement. To get two's complement:

1. Flip all the bits
2. Add 1 to the number

Problem 7 • *Write down all possible 4-digit binary numbers and corresponding decimal values if they are in two's complement format? What is the minimum and maximum value?*

- *What is the minimum and maximum value of n-digit signed binary number in two's complement?*

Problem 8 *Determine the decimal values of the following 1's complement 6-digit binary numbers :*

1. *01101110*
2. *10101101*

Problem 9 *Determine the decimal values of the following 2's complement 6-digit numbers :*

1. *01011110*
2. *10010111*

Problem 10 *Convert the decimal numbers 73, 23, -17, and -163 into signed 8-bit numbers in the following representations:*

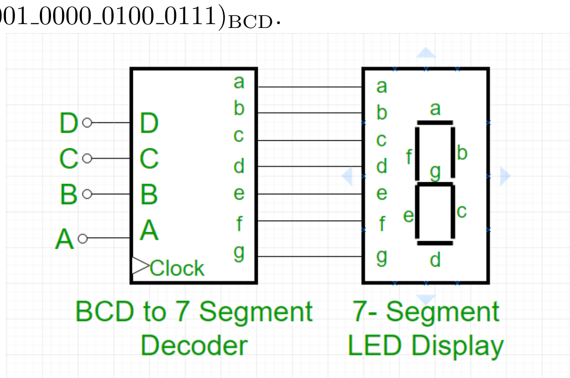
1. *Sign and magnitude*
2. *1's complement*
3. *2's complement*

7 Addition and subtraction of different signed representations

Problem 11 *Convert the decimal numbers -17 and +23 into the three different representations of 6-digit signed binary numbers and try adding them. What adjustments will you need to make to get the right result's (23-17=6) binary representation.*

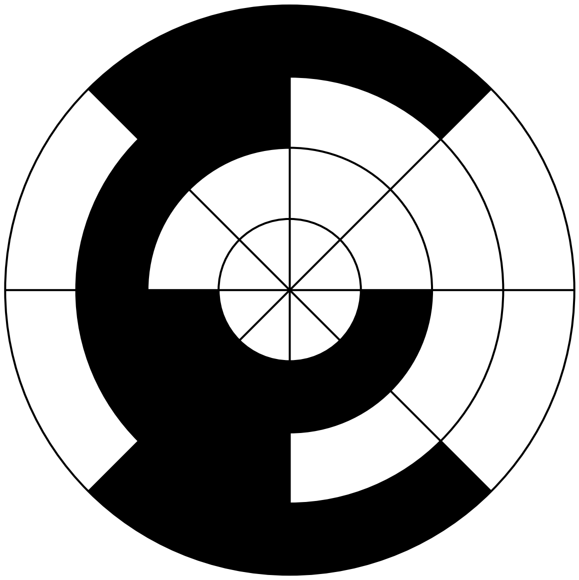
8 Binary coded decimal

In Binary coded decimal (BCD), each decimal digit is represented by 4 bits. For example, $1047 = (0001\_0000\_0100\_0111)_{\text{BCD}}$.



9 Gray code

A sequence of binary numbers that only change by 1 bit with increment of 1.



Problem 12 *Write all possible 3-bit binary numbers in gray-code*