Homework 1

Max marks: 80

Due on September 10, 2021, before class.

Problem 1 *Use algebraic manipulation to find the minimum sum-of-products expression for the function $f = x_1x_3 + x_1\bar{x}_2 + \bar{x}_1x_2x_3 + \bar{x}_1\bar{x}_2\bar{x}_3$. [1, Prob 2.12][10 marks]*

Solution

$$f = x_1x_3 + x_1\bar{x}_2 + \bar{x}_1x_2x_3 + \bar{x}_1\bar{x}_2\bar{x}_3 \quad (1)$$

To find the minimum sum-of-products expression, we will fill the K-map step by step. f has 4 terms. We will fill K-map for each term separately and then join them together to get the full K-map.

1. K-map for the term x_1x_3 is,

	$\bar{x}_1$		x_1	
	$\bar{x}_2$	x_2		$\bar{x}_2$
$\bar{x}_3$	0	0	0	0
x_3	0	0	1	1

2. K-map for the term $x_1\bar{x}_2$ is:

	$\bar{x}_1$		x_1	
	$\bar{x}_2$	x_2		$\bar{x}_2$
$\bar{x}_3$	0	0	0	1
x_3	0	0	0	1

3. K-map for the term $\bar{x}_1x_2x_3$ is:

	$\bar{x}_1$		x_1	
	$\bar{x}_2$	x_2		$\bar{x}_2$
$\bar{x}_3$	0	0	0	0
x_3	0	1	0	0

4. K-map for the term $\bar{x}_1\bar{x}_2\bar{x}_3$ is:

	$\bar{x}_1$		x_1	
	$\bar{x}_2$	x_2		$\bar{x}_2$
$\bar{x}_3$	1	0	0	0
x_3	0	0	0	0

Taking OR of the four K-maps, we get the K-map for $f = x_1x_3 + x_1\bar{x}_2 + \bar{x}_1x_2x_3 + \bar{x}_1\bar{x}_2\bar{x}_3$:

	$\bar{x}_1$		x_1	
	$\bar{x}_2$	x_2		$\bar{x}_2$
$\bar{x}_3$	1	0	0	1
x_3	0	1	1	1

This K-map can be decomposed into the sum of three K-maps with two minterms each:

1.

	$\bar{x}_1$		x_1	
	$\bar{x}_2$	x_2		$\bar{x}_2$
$\bar{x}_3$	0	0	0	0
x_3	0	1	1	0

,

which corresponds to the expression, x_2x_3.

2.

	$\bar{x}_1$		x_1	
	$\bar{x}_2$	x_2		$\bar{x}_2$
$\bar{x}_3$	0	0	0	1
x_3	0	0	0	1

,

which corresponds to the expression, $x_1\bar{x}_2$.

3.

	$\bar{x}_1$		x_1	
	$\bar{x}_2$	x_2		$\bar{x}_2$
$\bar{x}_3$	1	0	0	1
x_3	0	0	0	0

,

which corresponds to the expression, $\bar{x}_2\bar{x}_3$.

Taking the OR of previous 3 K-maps, we get the minimum SOP expression for

$$f = x_2x_3 + x_1\bar{x}_2 + \bar{x}_2\bar{x}_3 \quad (2)$$

However, this problem specifically asks for finding the minimal sum-of-products expression *using algebraic manipulation.*

From the K-map solution, we note that the first expression x_2x_3 in (2) corresponds to the region in K-map that comes from x_1x_3 and $\bar{x}_1x_2x_3$ in (1). This means we should be able to write x_2x_3 from $x_1x_3 + \bar{x}_1x_2x_3$.

$$\begin{aligned}
&x_1x_3 + \bar{x}_1x_2x_3 \\
&= x_1 \cdot 1 \cdot x_3 + \bar{x}_1x_2x_3 && \because 1 \cdot z = z \\
&= x_1(x_2 + \bar{x}_2)x_3 + \bar{x}_1x_2x_3 && \because 1 = z + \bar{z} \\
&= x_1x_2x_3 + x_1\bar{x}_2x_3 + \bar{x}_1x_2x_3 && \text{dist. prop.} \\
&= (x_1 + \bar{x}_1)x_2x_3 + x_1\bar{x}_2x_3 && \text{dist. prop.} \\
&= x_2x_3 + x_1\bar{x}_2x_3 && \because z + \bar{z} = 1
\end{aligned} \tag{3}$$

From the K-map solution, we also note that second term $x_1\bar{x}_2$ need not change. Although it can be combined with $\bar{x}_1\bar{x}_2\bar{x}_3$ to get the third term $\bar{x}_2\bar{x}_3$ of the (2).

$$\begin{aligned}
&x_1\bar{x}_2 + \bar{x}_1\bar{x}_2\bar{x}_3 \\
&= x_1\bar{x}_2(1 + \bar{x}_3) + \bar{x}_1\bar{x}_2\bar{x}_3 && \because 1 + z = 1 \\
&= x_1\bar{x}_2 + x_1\bar{x}_2\bar{x}_3 + \bar{x}_1\bar{x}_2\bar{x}_3 && \text{dist. prop.} \\
&= x_1\bar{x}_2 + (x_1 + \bar{x}_1)\bar{x}_2\bar{x}_3 && \text{dist. prop.} \\
&= x_1\bar{x}_2 + \bar{x}_2\bar{x}_3
\end{aligned} \tag{4}$$

Taking OR of (3) and (4), we get f on the LHS, but on the RHS we have an additional term of $x_1\bar{x}_2x_3$,

$$f = x_2x_3 + x_1\bar{x}_2x_3 + x_1\bar{x}_2 + \bar{x}_2\bar{x}_3. \tag{5}$$

That is not a problem, because $x_1\bar{x}_3x_3$ can be easily *absorbed* by $x_1\bar{x}_2$,

$$\begin{aligned}
&x_1\bar{x}_2x_3 + x_1\bar{x}_2 \\
&= x_1\bar{x}_2(x_3 + 1) && \text{dist. prop.} \\
&= x_1\bar{x}_2 && \because 1 + z = 1
\end{aligned} \tag{6}$$

Putting (6) in (5), we get the desired simplest (in terms of number of inputs and number of gates) SOP expression by *algebraic manipulation*,

$$f = x_2x_3 + x_1\bar{x}_2 + \bar{x}_2\bar{x}_3. \tag{7}$$

Problem 2 *Use algebraic manipulation to find the minimum sum-of-products expression for the function* $f = x_1\bar{x}_2\bar{x}_3 + x_1x_2x_4 + x_1\bar{x}_2x_3\bar{x}_4$. *[1, Prob 2.13][10 marks]*

Solution

Let's try another approach to illustrate this. This time let's color the K-maps according the product terms. Let's assign a color to each of the terms,

$$f = \color{red}{x_1\bar{x}_2\bar{x}_3} + \color{blue}{x_1x_2x_4} + \color{green}{x_1\bar{x}_2x_3\bar{x}_4}. \tag{8}$$

This expression has 4-variables, so we need $2^4 = 16$ cell K-map,

		$\bar{x}_1$		x_1	
		$\bar{x}_2$	x_2		$\bar{x}_2$
$\bar{x}_3$	$\bar{x}_4$	0	0	0	$\color{red}{1}$
	x_4	0	0	$\color{blue}{1}$	$\color{red}{1}$
x_3	x_4	0	0	$\color{blue}{1}$	0
	$\bar{x}_4$	0	0	0	$\color{green}{1}$

.

The ones in the K-map are already paired-up except the green $\color{green}{1}$, which we can pair up with one of the top red $\color{red}{1}$.

		$\bar{x}_1$		x_1	
		$\bar{x}_2$	x_2		$\bar{x}_2$
$\bar{x}_3$	$\bar{x}_4$	0	0	0	$\color{green}{1}$+$\color{red}{1}$
	x_4	0	0	$\color{blue}{1}$	$\color{red}{1}$
x_3	x_4	0	0	$\color{blue}{1}$	0
	$\bar{x}_4$	0	0	0	$\color{green}{1}$

.

Here we use $\color{green}{1}$+$\color{red}{1}$ to highlight that the minterm $x_1\bar{x}_2\bar{x}_3\bar{x}_4$ is paired up with two terms: green and red. Now, we can read the K-map into an expression,

$$f = \color{red}{x_1\bar{x}_2\bar{x}_3} + \color{blue}{x_1x_2x_4} + \color{green}{x_1\bar{x}_2\bar{x}_4} \tag{9}$$

To derive the same result from *algebraic manipulation*, it is clear that we only need to manipulate the red and green terms; the blue term stays untouched.

$$\begin{aligned}
&\color{red}{x_1\bar{x}_2\bar{x}_3} + \color{green}{x_1\bar{x}_2x_3\bar{x}_4} \\
&= \color{red}{x_1\bar{x}_2\bar{x}_3(1 + \bar{x}_4)} + \color{green}{x_1\bar{x}_2x_3\bar{x}_4} && \because 1 = 1 + z \\
&= \color{red}{x_1\bar{x}_2\bar{x}_3} + x_1\bar{x}_2\bar{x}_3\bar{x}_4 + \color{green}{x_1\bar{x}_2x_3\bar{x}_4} && \text{dist. prop.} \\
&= \color{red}{x_1\bar{x}_2\bar{x}_3} + \color{green}{x_1\bar{x}_2(\bar{x}_3 + x_3)\bar{x}_4} && \text{dist. prop..} \\
&= \color{red}{x_1\bar{x}_2\bar{x}_3} + \color{green}{x_1\bar{x}_2\bar{x}_4} && \because z + \bar{z} = 1
\end{aligned} \tag{10}$$

Putting (10) in (8), we get,

$$f = \color{red}{x_1\bar{x}_2\bar{x}_3} + \color{blue}{x_1x_2x_4} + \color{green}{x_1\bar{x}_2\bar{x}_4}, \tag{11}$$

which is the simplest SOP expression.

Problem 3 *Draw a timing diagram for the circuit in Figure 1. Show the waveforms that can be observed on all wires (f, g, h, k, l) in the circuit.[1, Prob 2.8][10 marks]*

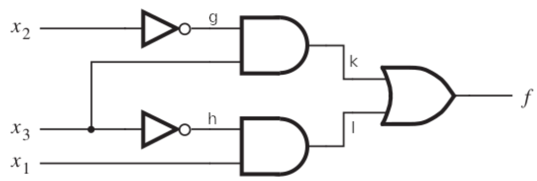
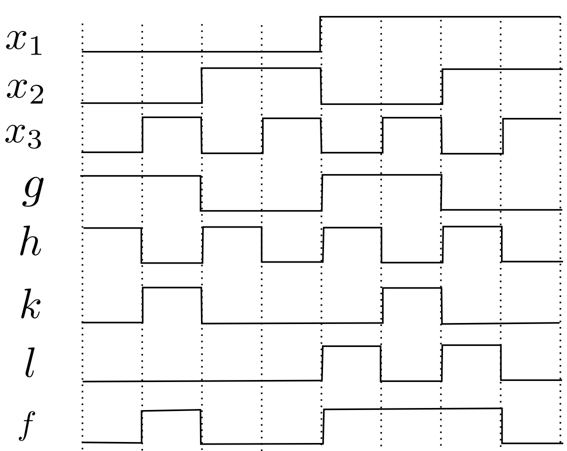


Figure 1: A three-input circuit

Row number	x_1	x_2	x_3	$f(x_1, x_2, x_3)$
0	0	0	0	0
1	0	0	1	1
2	0	1	0	0
3	0	1	1	0
4	1	0	0	1
5	1	0	1	1
6	1	1	0	1
7	1	1	1	0

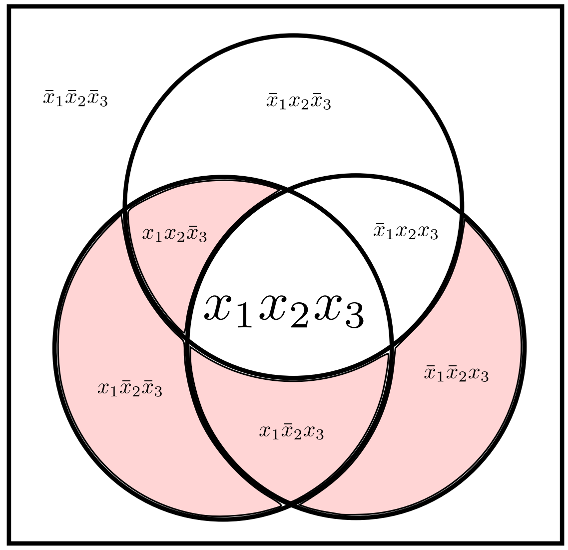
Figure 2: A three-variable function

Solution



Problem 4 *Represent the function in Figure 2 in the form of a Venn diagram and find its minimal sum-of-products form. [1, Prob 2.17][10 marks]*

Solution



Minimal SOP form is,

$$f = x_1\bar{x}_3 + \bar{x}_2 x_3$$

Problem 5 *Use algebraic manipulation to prove that $(x+y)\cdot(x+\bar{y}) = x$. [1, Prob 2.2] [10 marks].*

Solution

$$\begin{aligned}
\text{LHS} &= (x+y)cdot(x+\bar{y}) \\
&= (x+y)x + (x+y)\bar{y} && \text{dist. prop.} \\
&= x\cdot x + yx + x\bar{y} + y\cdot\bar{y} && \text{dist. prop.} \\
&= x + yx + x\bar{y} + 0 && \because y\cdot\bar{y} = 0 \\
&= x(1 + y + \bar{y}) && \text{dist. prop.} \\
&= x\cdot 1 && \because 1 + z = 1 \\
&= x = \text{RHS} && \because x\cdot 1 = x
\end{aligned} \tag{12}$$

Problem 6 *Determine whether or not the following expressions are valid, i.e., whether the left- and right-hand sides represent the same function. [1, Prob 2.7][10 marks]*

1. $x_1\bar{x}_3 + x_2x_3 + \bar{x}_2\bar{x}_3 = (\bar{x}_1 + \bar{x}_2 + x_3)(x_1 + x_2 + \bar{x}_3)(\bar{x}_1 + x_2 + \bar{x}_3)$

2. $(x_1 + x_3)(\bar{x}_1 + \bar{x}_2 + \bar{x}_3)(\bar{x}_1 + x_2) = (x_1 + x_2)(x_2 + x_3)(\bar{x}_1 + \bar{x}_3)$

Solution 6.1

$$\begin{aligned}\text{LHS} &= x_1\bar{x}_3 + x_2x_3 + \bar{x}_2\bar{x}_3\\ &= x_1(x_2+\bar{x}_2)\bar{x}_3 + (x_1+\bar{x}_1)x_2x_3\\ &\quad + (x_1+\bar{x}_1)\bar{x}_2\bar{x}_3\\ &= x_1x_2\bar{x}_3 + x_1\bar{x}_2\bar{x}_3 + x_1x_2x_3 + \bar{x}_1x_2x_3\\ &\quad + x_1\bar{x}_2\bar{x}_3 + \bar{x}_1\bar{x}_2\bar{x}_3\\ &= \sum m(6,4,7,3,4,0) = \sum m(0,3,4,6,7)\end{aligned}$$

$$\begin{aligned}\text{RHS} &= (\bar{x}_1+\bar{x}_2+x_3)(x_1+x_2+\bar{x}_3)(\bar{x}_1+x_2+\bar{x}_3)\\ &= \prod M(6,1,5)\\ &= \sum m(0,2,3,4,7)\end{aligned}$$

Since LHS $\neq$ RHS, hence the expression (1) is not valid.

Solution 6.2

Take the inversion of both sides of the equation, (2) is valid if and only if

$$\begin{aligned}&\overline{(x_1+x_3)(\bar{x}_1+\bar{x}_2+\bar{x}_3)(\bar{x}_1+x_2)}\\ &\quad = \overline{(x_1+x_2)(x_2+x_3)(\bar{x}_1+\bar{x}_3)}.\\ &\text{or } \bar{x}_1\bar{x}_3 + x_1x_2x_3 + x_1\bar{x}_2 = \bar{x}_1\bar{x}_2 + \bar{x}_2\bar{x}_3 + x_1x_3\end{aligned}$$

$$\begin{aligned}\text{LHS} &= \bar{x}_1\bar{x}_3 + x_1x_2x_3 + x_1\bar{x}_2\\ &= \bar{x}_1(x_2+\bar{x}_2)\bar{x}_3 + x_1x_2x_3 + x_1\bar{x}_2(x_3+\bar{x}_3)\\ &= \bar{x}_1x_2\bar{x}_3 + \bar{x}_1\bar{x}_2\bar{x}_3 + x_1x_2x_3 + x_1\bar{x}_2x_3\\ &\quad + x_1\bar{x}_2\bar{x}_3\\ &= \sum m(2,0,7,5,4) = \sum m(0,2,4,5,7)\end{aligned}$$

$$\begin{aligned}\text{RHS} &= \bar{x}_1\bar{x}_2 + \bar{x}_2\bar{x}_3 + x_1x_3\\ &= \bar{x}_1\bar{x}_2(x_3+\bar{x}_3) + (x_1+\bar{x}_1)\bar{x}_2\bar{x}_3\\ &\quad + x_1(x_2+\bar{x}_2)x_3\\ &= \bar{x}_1\bar{x}_2x_3 + \bar{x}_1\bar{x}_2\bar{x}_3 + x_1\bar{x}_2\bar{x}_3 + \bar{x}_1\bar{x}_2\bar{x}_3\\ &\quad + x_1x_2x_3 + x_1\bar{x}_2x_3\\ &= \sum m(1,0,4,0,7,5)\end{aligned}$$

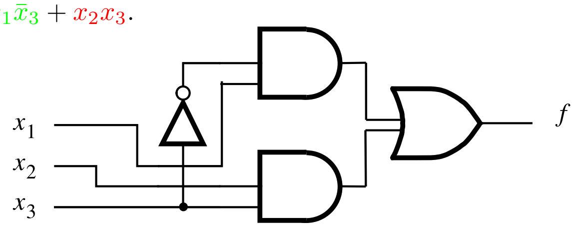
Since LHS $\neq$ RHS the expression is not valid.

Problem 7 *Design the simplest sum-of-products circuit that implements the function $f(x_1,x_2,x_3) = \sum m(3,4,6,7)$. [1, Prob 2.21][10 marks]*

Solution

	$\bar{x}_1$		x_1	
	$\bar{x}_2$	x_2		$\bar{x}_2$
$\bar{x}_3$	0	0	1	1
x_3	0	1	1	0

Simplest SOP expression is, $f(x_1,x_2,x_3) = x_1\bar{x}_3 + x_2x_3$.



Problem 8 *Design the simplest product-of-sums circuit that implements the function $f(x_1,x_2,x_3) = \prod M(0,2,5)$. [1, Prob 2.22][10 marks]*

Solution

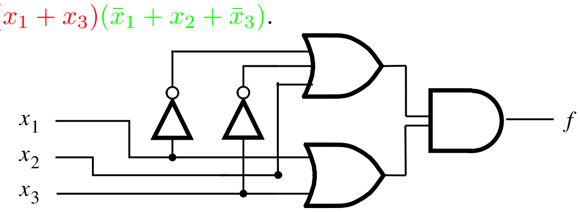
$$\bar{f}(x_1,x_2,x_3) = \sum m(0,2,5) \tag{13}$$

The K-map is for $\bar{f}$ is:

	$\bar{x}_1$		x_1	
	$\bar{x}_2$	x_2		$\bar{x}_2$
$\bar{x}_3$	1	1	0	0
x_3	0	0	0	1

Simplest SOP expression for $\bar{f}(x_1,x_2,x_3) = \bar{x}_1\bar{x}_3 + x_1\bar{x}_2x_3$.

By DeMorgan's theorem, we get the simplest POS expression is, $f(x_1,x_2,x_3) = (x_1+x_3)(\bar{x}_1+x_2+\bar{x}_3)$.



References

[1] S. Brown and Z. Vranesic. *Fundamentals of Digital Logic with Verilog Design: Third Edition*. McGraw-Hill Higher Education, 2013.