ECE 417 Midterm 1 2022

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Feb 18th, 2021

(1) Student name: Student email:

About the exam

1. There are total 5 problems. You must attempt all 5.
2. Maximum marks: 50 (70 with bonus marks).
3. Maximum time allotted: 50 min
4. Calculators are allowed.
5. One US Letter size or A4 size cheat sheet (both-sides) is allowed.

$\times$ $\hat{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\hat{y} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\hat{x} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$ $\hat{y} = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$

Problem 1 *Consider the basis matrix $B \in \mathbb{R}^{2\times 2}$ formed by mutually orthogonal unit-vectors $\hat{\mathbf{x}} \in \mathbb{R}^2$ and $\hat{\mathbf{y}} \in \mathbb{R}^2$, $B = [\hat{\mathbf{x}}, \hat{\mathbf{y}}]$. Prove that $B^\top B = I$. What is B^{-1}? (5 min, 5 marks)*

✓ $\hat{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ $\hat{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$

Knowledge required

① Block wise matrix transpose

$$M = \left[\begin{array}{cc|c} 1 & 2 & 3 \\ 2 & 5 & 6 \\ \hline 7 & 8 & 9 \end{array}\right] = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \qquad M^T = \left[\begin{array}{cc|c} 1 & 4 & 7 \\ 2 & 5 & 8 \\ \hline 3 & 6 & 9 \end{array}\right]$$

A^T, C^T, B^T, D^T

$$M^T = \begin{bmatrix} A^T & C^T \\ B^T & D^T \end{bmatrix}$$

② Block wise matrix multiplication

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} P & Q \\ R & S \end{bmatrix} = \begin{bmatrix} AP + BR & AQ + BS \\ CP + DR & CQ + DS \end{bmatrix}$$

$$B = \begin{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} & \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \end{bmatrix} \qquad B^T = \begin{bmatrix} x_1 & x_2 \\ y_1 & y_2 \end{bmatrix}$$

$$LHS = B^T B = \underbrace{\begin{bmatrix} \hat{\underline{x}} & \hat{\underline{y}} \end{bmatrix}^T}\begin{bmatrix} \hat{\underline{x}} & \hat{\underline{y}} \end{bmatrix} = \begin{bmatrix} \hat{\underline{x}}^T \\ \underline{y}^T \end{bmatrix} \begin{bmatrix} \hat{\underline{x}} & \hat{\underline{y}} \end{bmatrix}$$

Check whether blocks are compatible for matrix mul

$1 \times n$ $n \times 1$

$\rightarrow 1 \times 1$

$$= \begin{bmatrix} \hat{\underline{x}}^T \hat{\underline{x}} & \hat{\underline{x}}^T \underline{y} \\ \underline{y}^T \hat{\underline{x}} & \underline{y}^T \underline{y} \end{bmatrix}$$

$\hat{x}^T \hat{x} = 1$

$\hat{y}^T \hat{y} = 1$

Unit vector $\underline{a} \Rightarrow \|\underline{a}\| = 1$

$$\|\underline{a}\| = \sqrt{\underline{a}^T \underline{a}}$$

$$\|\underline{a}\| = \sqrt{a_1^2 + a_2^2 + \cdots a_n^2}$$

$$[a_1 \ \cdots \ a_n] \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} = a_1^2 + a_2^2 + \cdots + a_n^2$$

$$\sqrt{\underline{a}^T \underline{a}} = 1 \Rightarrow \underline{a}^T \underline{a} = 1$$

Orthogonality $\quad \underline{a} \cdot \underline{b} = 0 \Rightarrow \underline{a}^T \underline{b} = \underline{b}^T \underline{a} = 0$

$\hat{\underline{x}}, \hat{\underline{y}} \Rightarrow \hat{\underline{x}}^T \hat{\underline{y}} = \underline{y}^T \underline{x} = 0$

$$L.H.S = B^T B = \begin{bmatrix} \hat{\underline{x}}^T \underline{x} & \hat{\underline{x}} \cdot \underline{y} \\ \hat{\underline{y}}^T \underline{x} & \hat{\underline{y}}^T \underline{y} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I = R.H.S.$$

By defn of Inverse $\quad B^{-1} : \ B^{-1} B = B B^{-1} = I \Rightarrow B^{-1} = B^T$

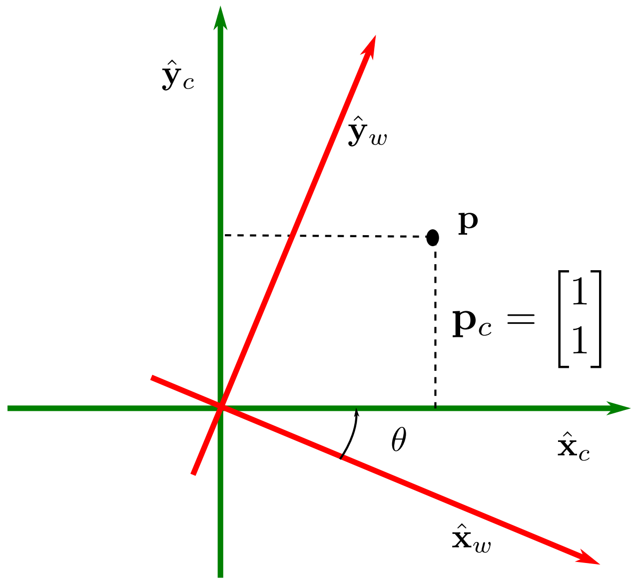


Figure 1: Rotation between red frame and green coordinate frame.

Problem 2 *For Fig 1, write the rotation matrix $R(\theta)$ that converts between coordinates of point from ~~red~~green coordinate frame $\mathbf{p}_c$ to ~~gren~~red coordinate frame $\mathbf{p}_w$ such that $\mathbf{p}_w = R(\theta)\mathbf{p}_c$. (Optional part) Test your formula if it looks correct for $\theta = 30^\circ$, $\mathbf{p}_c = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. (5 min, 5 marks)*

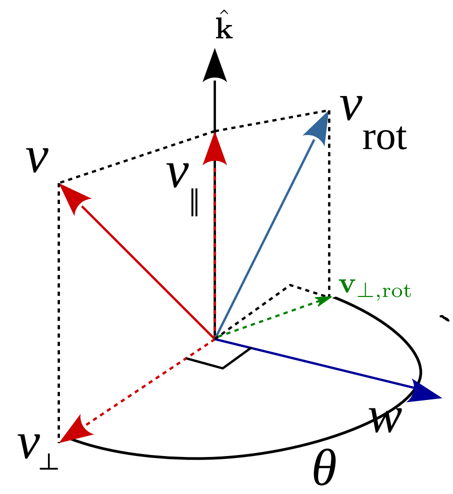
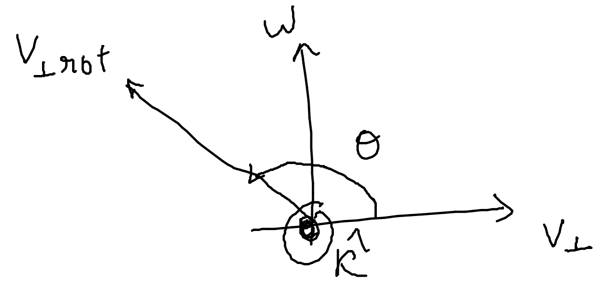
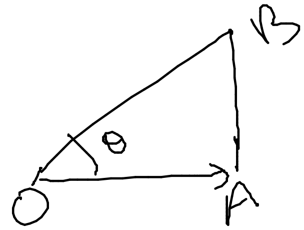
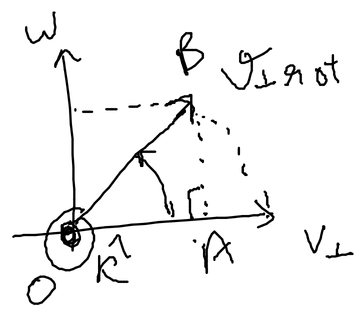


Figure 2: Rotation of point $\mathbf{v}$ around axis $\mathbf{k}$ by angle θ

Problem 3 *In Figure 2, we are rotating point* $\mathbf{v}$ *around axis unit-vector* $\hat{\mathbf{k}}$ *by an angle* θ*. We know that* $\mathbf{v}_{\perp} = -\hat{\mathbf{k}} \times (\hat{\mathbf{k}} \times \mathbf{v})$ *and* $\mathbf{w} = \hat{\mathbf{k}} \times \mathbf{v}$*. We get* $\mathbf{v}_{\perp,rot}$ *when we rotate* $\mathbf{v}_{\perp}$ *by angle* θ *around* $\hat{\mathbf{k}}$*. The diagram shows* θ *to be an obtuse angle, but your formula should be correct even when* θ *is an acute angle. First, write the value of in* $\mathbf{v}_{\perp,rot}$ *terms of* $\mathbf{v}_{\perp}$*,* $\mathbf{w}$ *and* θ*. Then write* $\mathbf{v}_{rot}$ *in terms of* $\mathbf{v}_{\parallel}$ *and* $\mathbf{v}_{\perp,rot}$*. (10 min, 10 marks)*







$$V_{\perp rot} = \vec{OB} = \vec{OA} + \vec{AB}$$

$$|OB| = \|V_\perp\| = \|V_{\perp rot}\|$$

$$|OA| = |OB| \cos\theta = \|V_\perp\| \cos\theta$$

$$|AB| = |OB| \sin\theta = \|V_\perp\| \sin\theta$$

$$V_{\perp rot} = \|OA\| \frac{v_\perp}{\|v_\perp\|} + |AB| \frac{w}{\|w\|}$$

$$= v_\perp \cos\theta + \frac{\|v_\perp\|\, w}{\|w\|} \sin\theta \quad] \checkmark$$

$$= v_\perp \cos\theta + w \sin\theta$$

$$\frac{\partial f(y)}{\partial y} = 0 \Rightarrow y = \_\ \_\ \_$$

$$y = x - p$$
$$x = y + p$$

Problem 4 *Using the fact that at minima (or maxima) of a differentiable function $f(\mathbf{x})$, $\nabla_{\mathbf{x}} f(\mathbf{x}) = 0$, find the minima of the following function,*

$$y = x + p$$

$$f(y) \qquad f(\mathbf{x}) = (\overbrace{\mathbf{x} - \mathbf{p}}^{y})^\top A^\top A(\overbrace{\mathbf{x} - \mathbf{p}}^{y}) + 2\mathbf{b}^\top \mathbf{x} + c, \tag{1}$$

where $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{p} \in \mathbb{R}^n$, $\mathbf{b} \in \mathbb{R}^n$, $c \in \mathbb{R}$, $A \in \mathbb{R}^{m \times n}$. Assume $A^\top A$ to be invertible. (10 min, 10 marks)

$$f(x) = (x - p)^T A^T A (x - p) + 2b^T x + c$$
$$= (x^T - p^T) A^T A (x - p) + 2b^T x + c$$
$$= (x^T - p^T)(A^T A x - A^T A p) + 2b^T x + c$$
$$= x^T A^T A x - \underbrace{x^T A^T A p} - \underbrace{p^T A^T A x} + p^T A^T A p + 2b^T x + c$$

$$(1 \times n)(n \times m)(m \times n)(n \times 1)$$
$$1 \times 1$$

$$x^T A^T A p = (x^T A^T A p)^T = p^T A^T A x$$

$$(AB)^T = B^T A^T$$

$$= \underbrace{x^T A^T A x}_{\text{Quadratic}} \underbrace{- 2p^T A^T A x + 2b^T x}_{\text{linear}} + \underbrace{c + p^T A^T A p}_{\text{constant}}$$

$$\frac{\partial}{\partial x} x^T Q x = 2Qx \quad , \quad \frac{\partial}{\partial x} q^T x = q$$

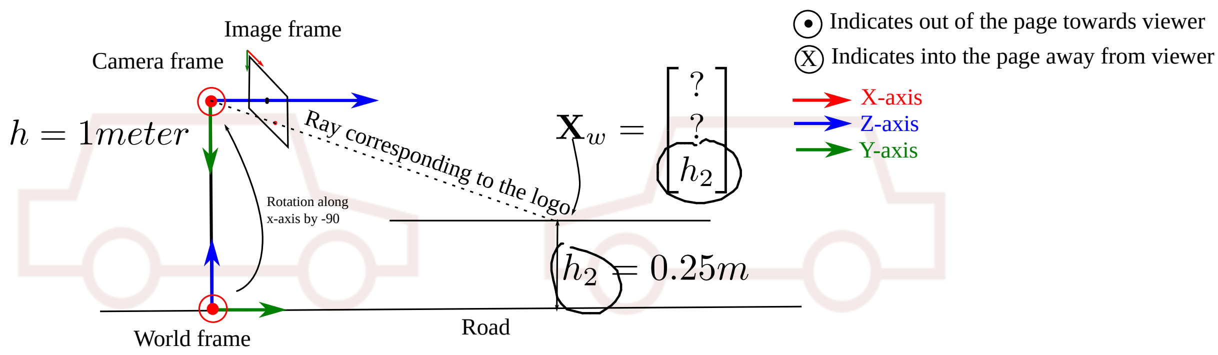
$$q^T = -2p^T A^T A + 2b^T$$
$$\Rightarrow q = -2A^T A p + 2b$$

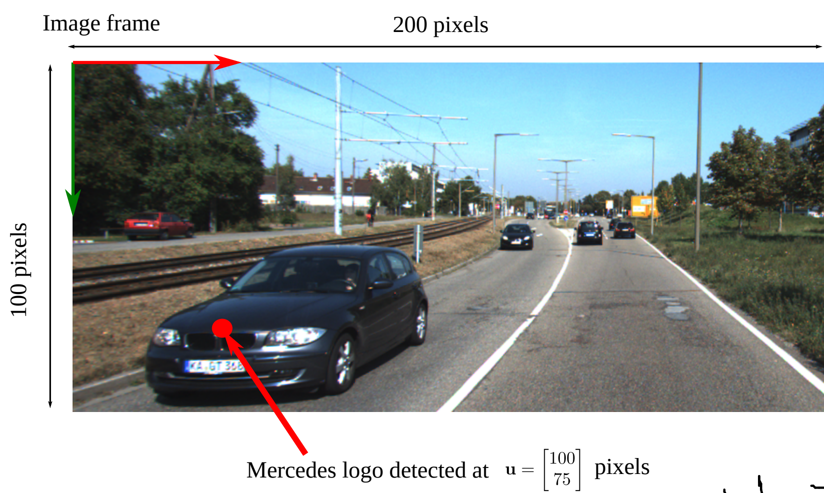
$$\frac{\partial f(x)}{\partial x} = 2A^T A x - 2A^T A p + 2b$$

$$\left.\frac{\partial f(x)}{\partial x}\right|_{x = x^*} = 0 \Rightarrow 2A^T A x - 2A^T A p + 2b = 0$$

$$\Rightarrow 2A^T A x = 2A^T A p - 2b$$

$$\Rightarrow x = (A^T A)^{-1} [A^T A p - b] = p - (A^T A)^{-1} b$$





$$K = \begin{bmatrix} 100 & 0 & 100 \\ 0 & 100 & 50 \\ 0 & 0 & 1 \end{bmatrix}$$

$$K^{-1} = \begin{bmatrix} 0.01 & 0 & -1 \\ 0 & 0.01 & -0.5 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\underline{u} = \begin{bmatrix} u \\ 1 \end{bmatrix} = \begin{bmatrix} 100 \\ 75 \\ 1 \end{bmatrix}$$

Figure 3: Image road triangulation

Problem 5 *In figure 4 find the 3D position of mercedes logo in the World coordinate frame, in terms of h (the height of the camera), image-coordinates of the mercedes logo $\mathbf{u}$, camera matrix K, and h_2 the height of logo from the road. The Camera mounted directly on top of the world frame. The road is a perfect plane and the pothole lies on the road plane (Equation of plane $Z_w = 0$ in the world coordinate frame). You do not need to substitute in the values, providing a formula or pseudo-code for computing the pothole coordinates is enough. (20 min, 20 marks, Bonus marks: 20)*

① $\lambda \underline{u} = K X_c$

$$X_c = K^{-1}(\lambda \underline{u}) = \lambda K^{-1} \underline{u}$$] eqn of ray

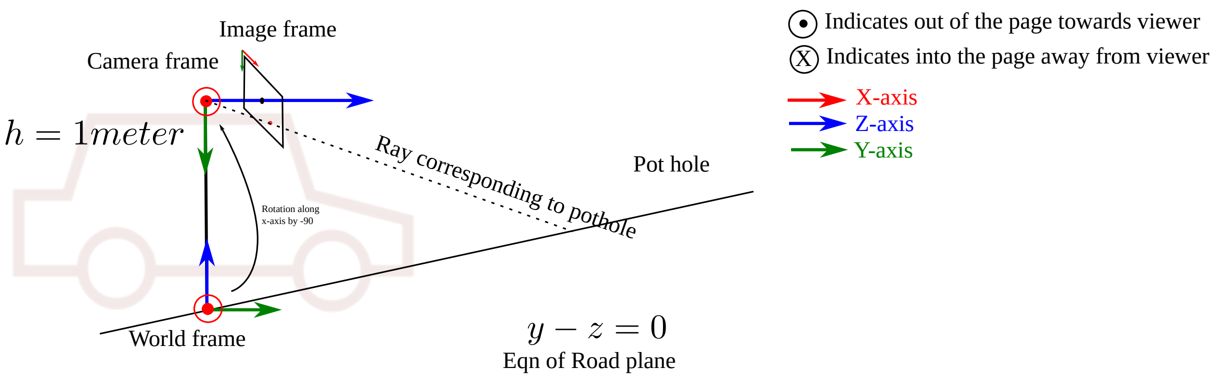
② $R = \begin{bmatrix} \quad \end{bmatrix}$ $t = \begin{bmatrix} 0 \\ 0 \\ h \end{bmatrix}$

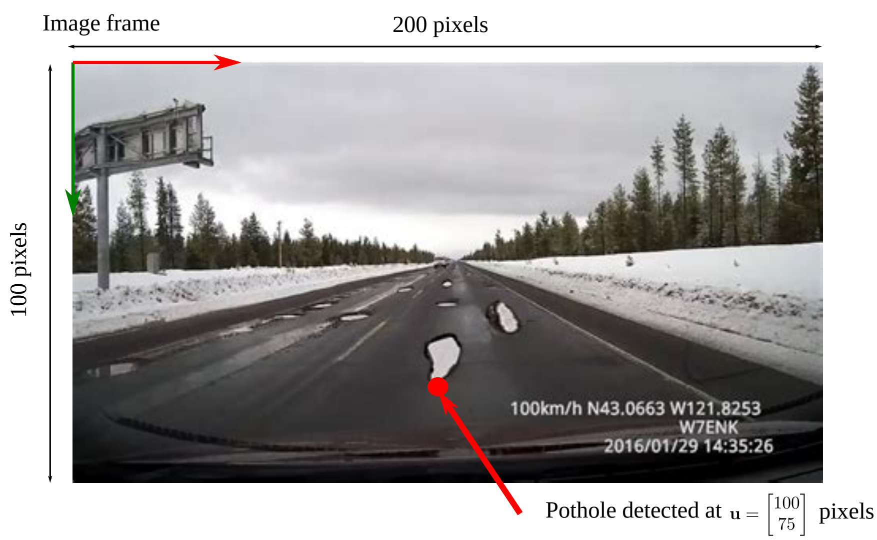
$$X_w = R X_c + t$$

$$\Rightarrow R K^{-1} \underline{u} + t \quad \Big] \text{ eqn of ray in wo}$$

③ $$(X_w)_3 = h_2$$

$$\lambda = \frac{h_2 - h}{[R K^{-1} \underline{u}]_3}$$





$$K = \begin{bmatrix} 100 & 0 & 100 \\ 0 & 100 & 50 \\ 0 & 0 & 1 \end{bmatrix}$$

Figure 4: Image road triangulation

Problem 6 *In figure 4 find the 3D position of the pothole in the World coordinate frame, in terms of h (the height of the camera), image-coordinates of the mercedes logo* $\mathbf{u}$*, camera matrix* K*. The Camera mounted directly on top of the world frame by rotating the world frame by* $90°$ *along X-axis. The road is a perfect plane and the pothole lies on the road plane whose equation in the world-coordinates is given by* $Y_w - Z_w = 1$*. You do not need to substitute in the values, providing a formula or pseudo-code for computing the pothole coordinates is enough. (20 min, 20 marks, Bonus marks: 20)*