

# ECE 417 Midterm 2 2022 practice problem set

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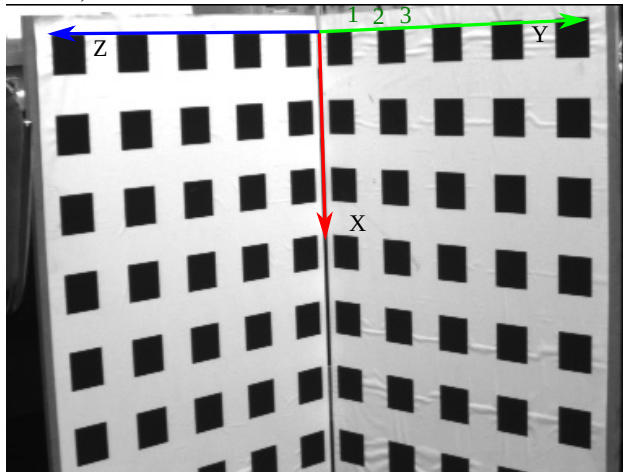
## **About the exam**

1. There are total 5 problems. You must attempt all 5.
2. Maximum marks: 50 (70 with bonus marks).
3. Maximum time allotted: 50 min
4. Calculators are allowed.
5. One US Letter size or A4 size cheat sheet (both-sides) is allowed.

**Problem 1** Given a set of  $n \geq 6$  points  $\underline{X}_i \in \mathbb{P}^3$  for all  $i \in \{1, \dots, n\}$  in 3D projective space, and a set of corresponding points  $\underline{u}_i \in \mathbb{P}^2$  in an image, find the 3D to 2D projective  $P \in \mathbb{R}^{3 \times 4}$  matrix that converts  $\underline{X}_i$  to  $\underline{u}_i = \lambda_i P \underline{X}_i$ . In other words, convert  $\underline{u}_i \times P \underline{X}_i = 0$  into a familiar form  $Ay = b$  or  $Ay = 0$  so that we can

solve for  $P$ . For notation purposes, you can denote  $\underline{u}_i = [x_i, y_i, w_i]^T$  and  $P = \begin{bmatrix} p_1^T \\ p_2^T \\ p_3^T \end{bmatrix}$  where  $p_1, p_2, p_3 \in \mathbb{R}^4$

are the rows of  $P$  represented as 4-D column vectors. (Practical motivation: We did camera calibration in lab using a single checker board. It is much easier to compute camera calibration using two mutually perpendicular checker boards so that all points do not lie on a single plane (hence linearly independent). One can make a coordinate system attached to the double checker and compute the 3D coordinates of each corner point in that system. Let  $\underline{X}_i \in \mathbb{P}^3$  be such points in 3D on the checker-board. Let  $\underline{u}_i \in \mathbb{P}^2$  be a point detected in the image so that we have one-to-one correspondence between  $\underline{X}_i$  and  $\underline{u}_i$ . Finding the projection matrix  $P \in \mathbb{R}^{3 \times 4}$  then reduces to the above problem. We will cover the breakdown of  $P$  matrix into  $P = K[R, t]$  in class.)



$$\underline{u}_i = K X_i \quad X_i = 3 \times 1$$

$$\underline{u}_i = K [R X_i + t]$$

$$\underline{u}_i = K \begin{bmatrix} R & t \\ & 1 \end{bmatrix} \begin{bmatrix} X_i \\ 1 \end{bmatrix}$$

$$X_i = 4 \times 1$$

$$P = K [R, t]$$

$$\underline{u}_i = \lambda P X_i$$

$$\underline{u}_i \times P X_i = 0$$

$P$  is  $3 \times 4$  matrix

$$P = \begin{bmatrix} p_1^T \\ p_2^T \\ p_3^T \end{bmatrix}$$

$p_1$  is  $4 \times 1$  vector

$p_2$   
 $p_3$

$$\underline{u}_i = \begin{bmatrix} 0 & -w_i & y_i \\ w_i & 0 & -x_i \\ -y_i & x_i & 0 \end{bmatrix}$$

$$A y = b$$

$$P X_i = \begin{bmatrix} p_1^T \\ p_2^T \\ p_3^T \end{bmatrix} X_i = \begin{bmatrix} p_1^T X_i \\ p_2^T X_i \\ p_3^T X_i \end{bmatrix}$$

$$P X_i = \begin{bmatrix} X_i^T p_1 \\ X_i^T p_2 \\ X_i^T p_3 \end{bmatrix}$$

$p_1$  is a 4x1  
 $X_i$  is a 4x1  
 $p_1^T$  is a 1x4

$p_1^T X_i$  is 1x1 scalar

$$u_i \times P X_i = 0$$

$$p_1^T X_i = (p_1^T X_i)^T = X_i^T p_1 = p_1^T X_i$$

$$\begin{bmatrix} 0 & -w_i & y_i \\ w_i & 0 & -x_i \\ -y_i & x_i & 0 \end{bmatrix} \begin{bmatrix} X_i^T p_1 \\ X_i^T p_2 \\ X_i^T p_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 - w_i X_i^T p_2 + y_i X_i^T p_3, \\ w_i X_i^T p_1 + 0 - x_i X_i^T p_3, \\ -y_i X_i^T p_1 + x_i X_i^T p_2 + 0 \end{bmatrix} = 0$$

$$\begin{bmatrix} 0^T & -w_i X_i^T & y_i X_i^T \\ w_i X_i^T & 0^T & -x_i X_i^T \\ -y_i X_i^T & x_i X_i^T & 0^T \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = 0$$

$\begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix}$  is a 12x1 vector

**Solution** Watch lecture <https://drive.google.com/file/d/1cY02DTagpckbY15gSOPYBu569v1ZUNN6/view?usp=sharing>

1. Write cross product as a matrix operation

$$[\underline{\mathbf{u}}_i]_{\times} = \begin{bmatrix} 0 & -w_i & y_i \\ w_i & 0 & -x_i \\ -y_i & x_i & 0 \end{bmatrix}$$

2. Write  $P\underline{\mathbf{X}}_i$  in terms of row vectors.

$$P\underline{\mathbf{X}}_i = \begin{bmatrix} \mathbf{p}_1^{\top} \\ \mathbf{p}_2^{\top} \\ \mathbf{p}_3^{\top} \end{bmatrix} \underline{\mathbf{X}}_i = \begin{bmatrix} \mathbf{p}_1^{\top} \underline{\mathbf{X}}_i \\ \mathbf{p}_2^{\top} \underline{\mathbf{X}}_i \\ \mathbf{p}_3^{\top} \underline{\mathbf{X}}_i \end{bmatrix}$$

3. Note that all the three terms like  $\mathbf{p}_1^{\top} \underline{\mathbf{X}}_i$  are scalars hence they are symmetric. Hence  $\mathbf{p}_1^{\top} \underline{\mathbf{X}}_i = \underline{\mathbf{X}}_i^{\top} \mathbf{p}_1$ .

$$P\underline{\mathbf{X}}_i = \begin{bmatrix} \underline{\mathbf{X}}_i^{\top} \mathbf{p}_1 \\ \underline{\mathbf{X}}_i^{\top} \mathbf{p}_2 \\ \underline{\mathbf{X}}_i^{\top} \mathbf{p}_3 \end{bmatrix}$$

4. Substitute these values in the original equation  $\underline{\mathbf{u}}_i \times P\underline{\mathbf{X}}_i = \mathbf{0}_{3 \times 1}$ .

$$\begin{bmatrix} 0 & -w_i & y_i \\ w_i & 0 & -x_i \\ -y_i & x_i & 0 \end{bmatrix} \begin{bmatrix} \underline{\mathbf{X}}_i^{\top} \mathbf{p}_1 \\ \underline{\mathbf{X}}_i^{\top} \mathbf{p}_2 \\ \underline{\mathbf{X}}_i^{\top} \mathbf{p}_3 \end{bmatrix} = \mathbf{0}_{3 \times 1}$$

5. Matrix multiply

$$\begin{bmatrix} 0 - w_i \underline{\mathbf{X}}_i^{\top} \mathbf{p}_2 + y_i \underline{\mathbf{X}}_i^{\top} \mathbf{p}_3 \\ w_i \underline{\mathbf{X}}_i^{\top} \mathbf{p}_1 + 0 - x_i \underline{\mathbf{X}}_i^{\top} \mathbf{p}_3 \\ -y_i \underline{\mathbf{X}}_i^{\top} \mathbf{p}_1 + x_i \underline{\mathbf{X}}_i^{\top} \mathbf{p}_2 + 0 \end{bmatrix} = \mathbf{0}_{3 \times 1}$$

6. Write the unknowns as a single vector, and the knowns as a matrix multiplication with the unknowns

$$\begin{bmatrix} \mathbf{0}^{\top} & -w_i \underline{\mathbf{X}}_i^{\top} & y_i \underline{\mathbf{X}}_i^{\top} \\ w_i \underline{\mathbf{X}}_i^{\top} & \mathbf{0}^{\top} & -x_i \underline{\mathbf{X}}_i^{\top} \\ -y_i \underline{\mathbf{X}}_i^{\top} & x_i \underline{\mathbf{X}}_i^{\top} & \mathbf{0}^{\top} \end{bmatrix}_{3 \times 12} \begin{bmatrix} \mathbf{p}_1 \\ \mathbf{p}_2 \\ \mathbf{p}_3 \end{bmatrix}_{12 \times 1} = \mathbf{0}_{3 \times 1}$$

7. Pick only two of the equations as only two are linearly independent.

$$\begin{bmatrix} \mathbf{0}^{\top} & -w_i \underline{\mathbf{X}}_i^{\top} & y_i \underline{\mathbf{X}}_i^{\top} \\ w_i \underline{\mathbf{X}}_i^{\top} & \mathbf{0}^{\top} & -x_i \underline{\mathbf{X}}_i^{\top} \end{bmatrix}_{2 \times 12} \begin{bmatrix} \mathbf{p}_1 \\ \mathbf{p}_2 \\ \mathbf{p}_3 \end{bmatrix}_{12 \times 1} = \mathbf{0}_{2 \times 1}$$

8. Collect all the equations from  $n$  pairs of corresponding points  $\underline{\mathbf{u}}_1, \dots, \underline{\mathbf{u}}_n$  and  $\underline{\mathbf{X}}_1, \dots, \underline{\mathbf{X}}_n$ .

$$\begin{bmatrix} \mathbf{0}^{\top} & -w_1 \underline{\mathbf{X}}_1^{\top} & y_1 \underline{\mathbf{X}}_1^{\top} \\ w_1 \underline{\mathbf{X}}_1^{\top} & \mathbf{0}^{\top} & -x_1 \underline{\mathbf{X}}_1^{\top} \\ \vdots & \vdots & \vdots \\ \mathbf{0}^{\top} & -w_n \underline{\mathbf{X}}_n^{\top} & y_n \underline{\mathbf{X}}_n^{\top} \\ w_n \underline{\mathbf{X}}_n^{\top} & \mathbf{0}^{\top} & -x_n \underline{\mathbf{X}}_n^{\top} \end{bmatrix}_{2n \times 12} \begin{bmatrix} \mathbf{p}_1 \\ \mathbf{p}_2 \\ \mathbf{p}_3 \end{bmatrix}_{12 \times 1} = \mathbf{0}_{2n \times 1}$$

9.  $P$  matrix has rank  $\text{rank}(P) = 11$  because it has 12 elements and equivalence upto a scale factor. So the solution of the above equation can be computed from SVD by choosing the right singular vector corresponding to the smallest singular value.

$$A = \begin{bmatrix} \mathbf{0}^\top & -w_1 \underline{\mathbf{X}}_1^\top & y_1 \underline{\mathbf{X}}_1^\top \\ w_1 \underline{\mathbf{X}}_1^\top & \mathbf{0}^\top & -x_1 \underline{\mathbf{X}}_1^\top \\ \vdots & \vdots & \vdots \\ \mathbf{0}^\top & -w_n \underline{\mathbf{X}}_n^\top & y_n \underline{\mathbf{X}}_n^\top \\ w_n \underline{\mathbf{X}}_n^\top & \mathbf{0}^\top & -x_n \underline{\mathbf{X}}_n^\top \end{bmatrix} = U \Sigma V^\top$$

Let  $V = [\mathbf{v}_1, \dots, \mathbf{v}_n]$ , then

$$\begin{bmatrix} \mathbf{p}_1 \\ \mathbf{p}_2 \\ \mathbf{p}_3 \end{bmatrix} = \mathbf{v}_n$$

Now we can write the  $P$  matrix as

$$P = \begin{bmatrix} \mathbf{p}_1^\top \\ \mathbf{p}_2^\top \\ \mathbf{p}_3^\top \end{bmatrix}$$

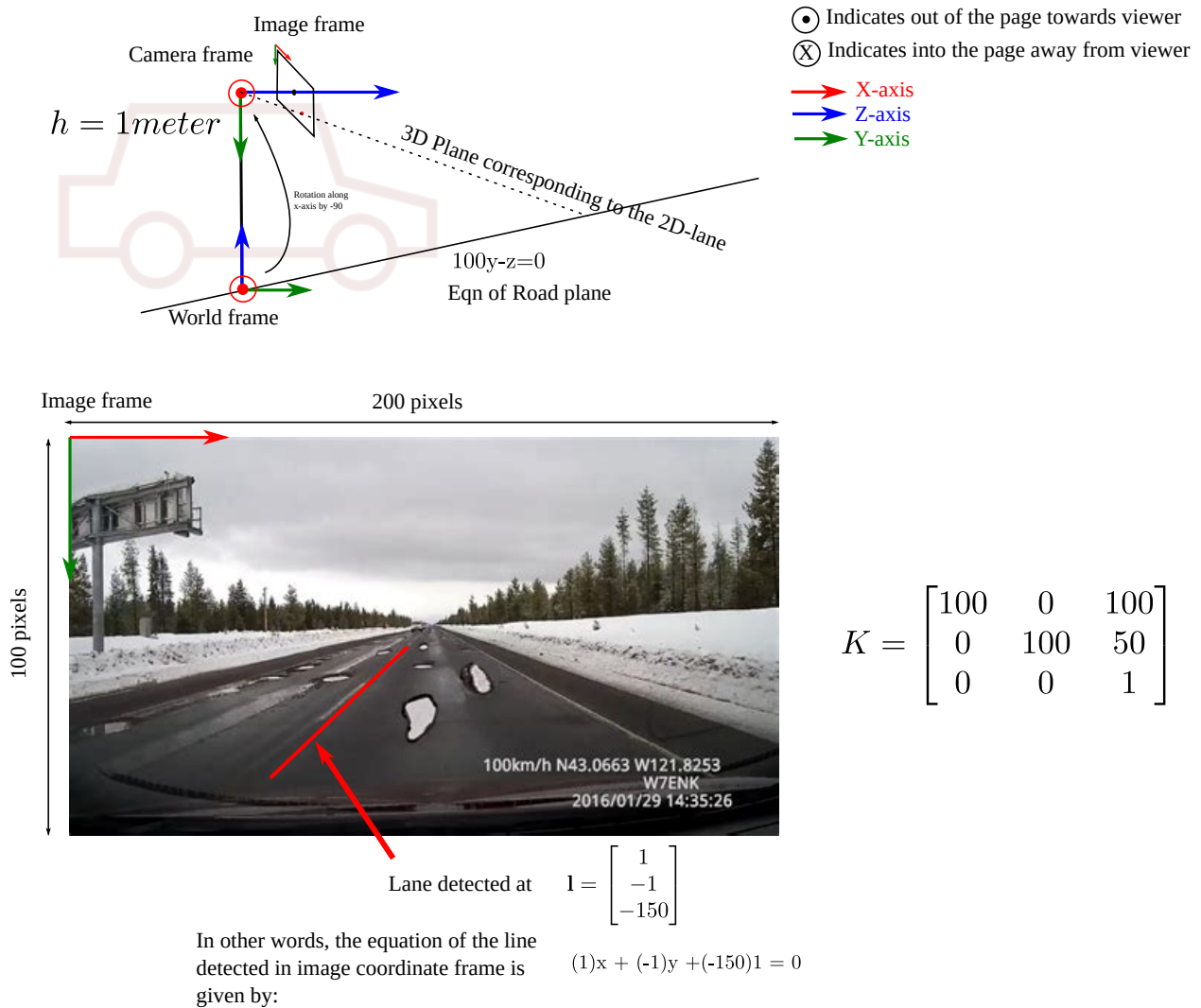


Figure 1: Line-plane triangulation

**Problem 2** In figure 1 find the 3D representation of the lane the World coordinate frame, in terms of  $h$  (the height of the camera), image-representation of the line  $\mathbf{l}$  (provided in figure), camera matrix  $K$  (provided in figure). Assume the lane to be a straight line. The Camera is mounted directly on top of the world frame, both of which are aligned to the gravity vector. The road is a perfect plane with a slope such that the equation of road plane in world-coordinate frame is given by  $100Y_w - Z_w = 0$  and the lane lies on the road plane. Provide the formula or pseudo-code for computing the 3D representation of the lane, and also substitute in the values. (20 min, 20 marks)

**Solution** Watch lecture [https://drive.google.com/file/d/1JaEwLxQ2BvT30sVxshmg1Bz\\_v5FjC1hW/view?usp=sharing](https://drive.google.com/file/d/1JaEwLxQ2BvT30sVxshmg1Bz_v5FjC1hW/view?usp=sharing) See homework 4 solution.



**Problem 3** Find the minimum point of the function,  $f(\mathbf{u}) = 2\mathbf{u}^\top A^\top A \mathbf{u} - 3\mathbf{u}^\top \mathbf{b} + 4\mathbf{c}^\top \mathbf{u} + d$ . Let  $\mathbf{u} \in \mathbb{R}^{n \times 1}$  be a  $n$ -dimensional vector and sizes of  $A, \mathbf{b}, \mathbf{c}, d$  be such that matrix multiplication and addition is valid. Also assume that  $A^\top A$  is full rank, hence invertible.

**Solution** Watch this lecture [https://drive.google.com/file/d/1wgY2LAW7LQnh\\_IyHY0XDAr2yorXta93Z/view?usp=sharing](https://drive.google.com/file/d/1wgY2LAW7LQnh_IyHY0XDAr2yorXta93Z/view?usp=sharing)

**Problem 4** Let matrix  $A \in \mathbb{R}^{m \times n}$  be a  $m \times n$  matrix. We are given that  $B = A^\top A$  has  $n$  orthonormal eigen vectors  $\mathbf{e}_1, \dots, \mathbf{e}_n$  with corresponding eigen values as  $\lambda_1 \dots \lambda_n$  such that  $B\mathbf{e}_i = \lambda_i \mathbf{e}_i$  for all  $i \in \{1, \dots, n\}$ . Let the rank of matrix  $A$  be  $r$ . Write the thin singular value decomposition of  $A = U_{m \times r} \Sigma_{r \times r} V_{n \times r}^\top$  in terms of eigen values and eigen vectors of matrix  $B = A^\top A$ .

**Solution** Go through these slides. Watch this lecture [https://drive.google.com/file/d/13a0\\_XI7kykQN0s5RJ0fhqtL5f/view?usp=sharing](https://drive.google.com/file/d/13a0_XI7kykQN0s5RJ0fhqtL5f/view?usp=sharing)

The matrix of right singular vectors of  $A$  is same as the eigen vector matrix of  $B = A^\top A$ .

$$V = [\mathbf{e}_1 \dots \mathbf{e}_r] \in \mathbb{R}^{n \times r} \quad (1)$$

The matrix of singular values are the square root of eigen values of  $B$ .

$$\Sigma = \begin{bmatrix} \sqrt{\lambda_1} & \dots & 0 \\ 0 & \ddots & 0 \\ 0 & \dots & \sqrt{\lambda_r} \end{bmatrix} \in \mathbb{R}^{r \times r} \quad (2)$$

$$U = [\mathbf{u}_1 \dots \mathbf{u}_r] \in \mathbb{R}^{m \times r} \quad (3)$$

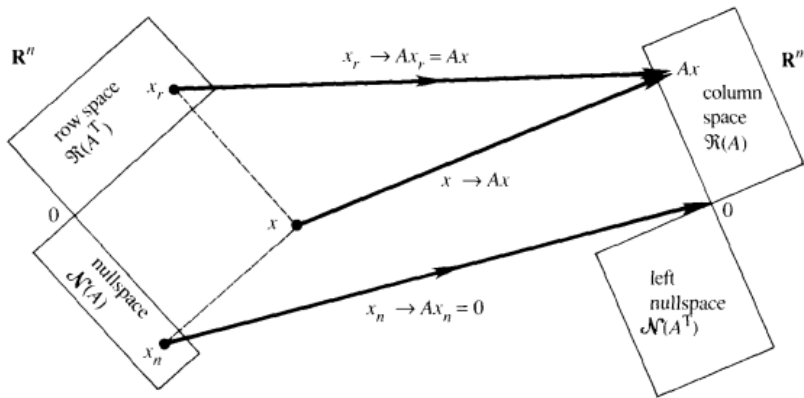
$$\text{where } \mathbf{u}_i = \frac{A\mathbf{e}_i}{\sqrt{\lambda_i}} \quad (4)$$



**Problem 5** Let matrix  $A \in \mathbb{R}^{m \times n}$  has the singular value decomposition (SVD) as  $A = U\Sigma V^T$  and rank of the matrix be  $r = \text{rank}(A)$ . Write the basis vectors of the four fundamental subspaces of matrix  $A$  in terms of SVD,

1. Null space of  $A$  ( $\mathcal{N}(A) = ?$ ).
2. Column space or range space ( $\mathcal{R}(A) = ?$ ).
3. Row space ( $\mathcal{R}(A^T) = ?$ ).
4. Left null space ( $\mathcal{N}(A^T) = ?$ ).

You can denote the first  $r$  column vectors of  $U$  as  $U_{1:r} \in \mathbb{R}^{m \times r}$  and the remaining  $m - r$  vectors as  $U_{r+1:m} \in \mathbb{R}^{m \times (m-r)}$ . Similarly for  $V$ , first  $r$  column vectors of  $V$  as  $V_{1:r} \in \mathbb{R}^{n \times r}$  and  $V_{r+1:n} \in \mathbb{R}^{n \times (n-r)}$ .



**Solution** Watch lecture <https://drive.google.com/file/d/17Cr0rMr567gFRNrrj9ri9vtsFaNLqW7t/view?usp=sharing>

1. Null space of  $A$  ( $\mathcal{N}(A) = V_{r+1:n}$ ).
2. Column space or range space ( $\mathcal{R}(A) = U_{1:r}$ ).
3. Row space ( $\mathcal{R}(A^\top) = V_{1:r}$ ).
4. Left null space ( $\mathcal{N}(A^\top) = U_{r+1:m}$ ).

## Extra practice problems

**Problem 6** Find a line passing through the following points

$$\mathbf{u}_1 = [101, 203]^\top, \mathbf{u}_2 = [49, 102]^\top, \mathbf{u}_3 = [27, 51]^\top, \mathbf{u}_4 = [201, 403]^\top, \mathbf{u}_5 = [74, 151]^\top.$$

You can leave the output in terms of SVD.

**Problem 7** Find a plane passing through the following points

$$\mathbf{x}_1 = [9.99, 101, 203]^\top, \mathbf{x}_2 = [5.1, 49, 102]^\top, \mathbf{x}_3 = [2.5, 27, 51]^\top, \mathbf{x}_4 = [21, 201, 403]^\top, \mathbf{x}_5 = [7.6, 74, 151]^\top.$$

You can leave the output in terms of SVD.

**Problem 8** Find the 3D line in parameteric representation that is formed by the intersection of two planes  $\mathbf{p}^\top \underline{\mathbf{x}} = 0$  (with  $\mathbf{p} = [1, 2, 3, 4]^\top$ ) and  $\mathbf{q}^\top \underline{\mathbf{x}} = 0$  where  $\mathbf{q} = [-3, 2, 1, 4]^\top$ .

**Problem 9** Find the point on the intersection of following 3D lines  $\mathbf{x} = \lambda_1 \mathbf{d}_1 + \mathbf{y}$  and  $\mathbf{x} = \lambda_2 \mathbf{d}_2 + \mathbf{z}$ . Here  $\lambda_1 \in \mathbb{R}$  and  $\lambda_2 \in \mathbb{R}$  are the free parameters. The rest of the parameters have the following values

$$\mathbf{d}_1 = [1, 2, 0]^\top, \mathbf{d}_2 = [-2, 1, 0]^\top, \mathbf{y} = [1, 2, 0]^\top, \mathbf{z} = [4, 5, 0]^\top$$

**Problem 10** Find the point of intersection of the 3D line  $\mathbf{x} = \lambda \mathbf{d} + \mathbf{x}_0$  with the 3D plane  $\mathbf{p}^\top \underline{\mathbf{x}} = 0$ . The parameters have the following

$$\mathbf{d} = [1, 2, 0]^\top, \mathbf{x}_0 = [3, 4, 5]^\top, \mathbf{p} = [1, 2, 0, 7]^\top$$

## Practice problem solutions

**Solution 6** Watch lecture [https://drive.google.com/file/d/13a0\\_XI7kykQN0s5RJ0fhqtL5fgEzVfkN/view?usp=sharing](https://drive.google.com/file/d/13a0_XI7kykQN0s5RJ0fhqtL5fgEzVfkN/view?usp=sharing) Let  $\mathbf{l} \in \mathbb{P}^2$  be the parameters of the line, so that  $\mathbf{u}^\top \mathbf{l} = 0$ .

$$A = \begin{bmatrix} \mathbf{u}_1^\top & 1 \\ \mathbf{u}_2^\top & 1 \\ \mathbf{u}_3^\top & 1 \\ \mathbf{u}_4^\top & 1 \\ \mathbf{u}_5^\top & 1 \end{bmatrix}_{5 \times 3}$$

We are looking for the solution of  $A\mathbf{l} = 0$ . Let the SVD of  $A = U\Sigma V^\top$ . Let  $V = [\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3]$ , then the representation of the line  $\mathbf{l} = \mathbf{v}_3$ .

**Solution 7** Watch the lecture <https://drive.google.com/file/d/1PEdgcHcF6Ud0WYmVtCR1sJfctDcnBm8E9/view?usp=sharing>

Let  $\mathbf{p} \in \mathbb{P}^3$  be the parameters of the plane, so that  $\mathbf{x}^\top \mathbf{p} = 0$ .

$$A = \begin{bmatrix} \mathbf{x}_1^\top & 1 \\ \mathbf{x}_2^\top & 1 \\ \mathbf{x}_3^\top & 1 \\ \mathbf{x}_4^\top & 1 \\ \mathbf{x}_5^\top & 1 \end{bmatrix}_{5 \times 4}$$

We are looking for the solution of  $A\mathbf{p} = 0$ . Let the SVD of  $A = U\Sigma V^\top$ . Let  $V = [\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4]$ , then the representation of the line  $\mathbf{p} = \mathbf{v}_4$ .

**Solution 8** Watch the lecture [https://drive.google.com/file/d/1JaEwLxQ2BvT30sVxshmg1Bz\\_v5FjC1hW/view?usp=sharing](https://drive.google.com/file/d/1JaEwLxQ2BvT30sVxshmg1Bz_v5FjC1hW/view?usp=sharing)  $\mathbf{x} = \lambda(\mathbf{p}_{1:3} \times \mathbf{q}_{1:3}) + \begin{bmatrix} \mathbf{p}_{1:3}^\top \\ \mathbf{q}_{1:3}^\top \end{bmatrix}^\dagger \begin{bmatrix} -p_4 \\ q_4 \end{bmatrix}$

**Solution 9** Watch the lecture <https://drive.google.com/file/d/1foVVQBC0kr1jjJ3f-UP4zshHn41-c611e/view?usp=sharing>

$$\begin{bmatrix} \mathbf{d}_1 & -\mathbf{d}_2 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = \mathbf{z} - \mathbf{y}$$

or

$$\begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} \mathbf{d}_1 & -\mathbf{d}_2 \end{bmatrix}^\dagger \mathbf{z} - \mathbf{y}$$

The point of intersection is given by

$$\mathbf{x} = \lambda_1 \mathbf{d}_1 + \mathbf{y}$$

**Solution 10** Watch the lecture [https://drive.google.com/file/d/1wgY2LAW7LQnh\\_IyHY0XDAR2yorXta93Z/view?usp=sharing](https://drive.google.com/file/d/1wgY2LAW7LQnh_IyHY0XDAR2yorXta93Z/view?usp=sharing)

$$\lambda \mathbf{p}_{1:3}^\top \mathbf{d} + \mathbf{p}_{1:3}^\top \mathbf{x}_0 + p_4 = 0$$

Solve for  $\lambda$ .

$$\lambda = -\frac{\mathbf{p}_{1:3}^\top \mathbf{x}_0 + p_4}{\mathbf{p}_{1:3}^\top \mathbf{d}}$$

Point of intersection is

$$\mathbf{x} = \lambda \mathbf{d} + \mathbf{x}_0$$



