ECE 417 Midterm 2 2022

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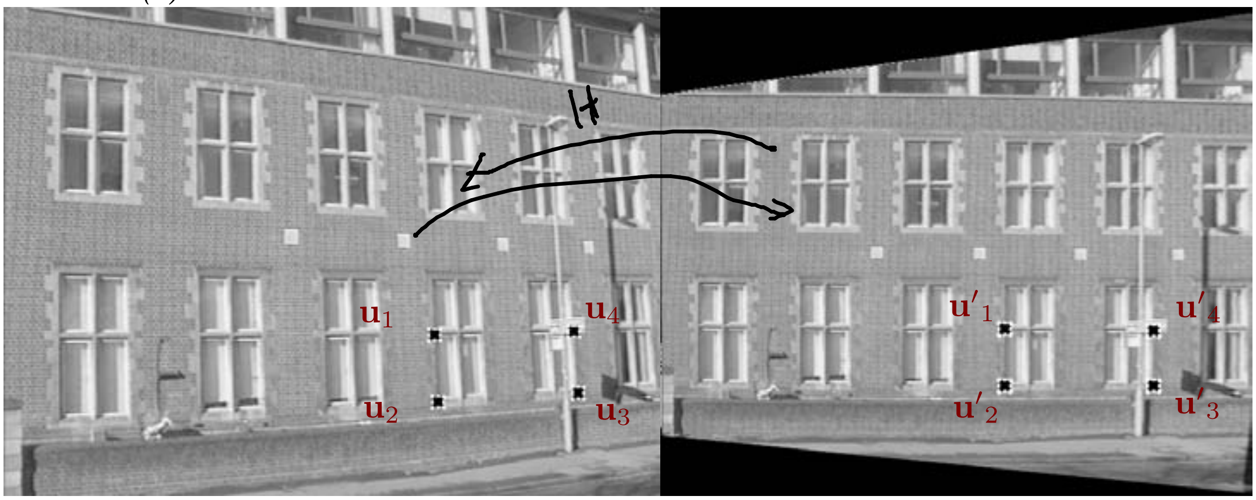
(1) Student name: Student email:

About the exam

1. There are total 5 problems. You must attempt all 5.
2. Maximum marks: 50 60 (55 65 with bonus marks).
3. Maximum time allotted: 50 min
4. Calculators are allowed.
5. One US Letter size or A4 size cheat sheet (both-sides) is allowed.
6. For all problems, you can assume that the singular value decomposition of any matrix is given and you can leave the answer in terms of SVD or formula without substituting in the values.

Problem 1 *Do either problem 1(a) or 1(b). (25 marks)*
Problem 1(a):



$\underline{u}'_i = \lambda_i H \underline{u}_i$

$i \in \{1,2,3,4\}$

$\underline{\mathbf{u}}_1 = [100, 98, 1]^\top$ $\quad$ $\underline{\mathbf{u}}_2 = [102, 95, 1]^\top$

$\underline{\mathbf{u}}_3 = [107, 90, 1]^\top$ $\quad$ $\underline{\mathbf{u}}_4 = [110, 85, 1]^\top$

$\underline{\mathbf{u}}'_1 = [100, 98, 1]^\top$ $\quad$ $\underline{\mathbf{u}}'_2 = [102, 95, 1]^\top$

$\underline{\mathbf{u}}'_3 = [107, 98, 1]^\top$ $\quad$ $\underline{\mathbf{u}}'_4 = [110, 95, 1]^\top$

Find H such that $\underline{\mathbf{u}}'_i = \lambda_i H \underline{\mathbf{u}}_i$ for any point on one image to another image, where $\underline{\mathbf{u}}'_i, \underline{\mathbf{u}}_i \in \mathbb{P}^2$ and $\lambda_i \in \mathbb{R}$ is a scalar. In other words, convert $\underline{\mathbf{u}}'_i \times H\underline{\mathbf{u}}_i = 0_{3\times1}$ into a system of linear equations that can be solved using familiar $A\mathbf{y} = 0$ form. For notation purposes, you can use $\underline{\mathbf{u}}_i \in \begin{bmatrix} x_i \\ y_i \\ w_i \end{bmatrix}$, $\underline{\mathbf{u}}'_i \in \begin{bmatrix} x'_i \\ y'_i \\ w'_i \end{bmatrix}$ and $H = \begin{bmatrix} \mathbf{h}_1^\top \\ \mathbf{h}_2^\top \\ \mathbf{h}_3^\top \end{bmatrix}$ where $\mathbf{h}_1, \mathbf{h}_2, \mathbf{h}_3 \in \mathbb{R}^3$. Cross product matrix of vector $\underline{\mathbf{u}}'_i$ is given as $\underline{\mathbf{u}}'_i\times = \begin{bmatrix} 0 & -w'_i & y'_i \\ w'_i & 0 & -x'_i \\ -y'_i & x'_i & 0 \end{bmatrix}$

$\underline{u}_i' \parallel H\underline{u}_i$

$T \quad \lambda_i H u_i$

u_i'

$\underbrace{\underline{u}_i'}_{3\times1} \times H\underline{u}_i = 0_{3\times1}$

$H \in \mathbb{R}^{3\times3}$

$\underline{u}_i \in \mathbb{R}^{3\times1}$

$\underline{a}_{3\times1} \times \underline{b}_{3\times1} = \underline{c}_{3\times1}$

$H\underline{u}_i \in \mathbb{R}^{3\times1}$

$\underline{u}_i' \times = \begin{bmatrix} \quad & \quad & \quad \end{bmatrix}_{3\times3} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}_{3\times}$

$$\begin{bmatrix} 0 & -w_i' & y_i' \\ w_i' & 0 & -x_i' \\ -y_i' & x_i' & 0 \end{bmatrix} H \underline{u}_i = 0_{3\times 1}$$

$$Ax = b$$

$$Ay = 0$$

$$H_{3\times 3} = \begin{bmatrix} \underline{h}_1^T \\ \underline{h}_2^T \\ \underline{h}_3^T \end{bmatrix}_{3\times 3}$$

$$\underline{h}_3 \; \underline{h}_2 \; \underline{h}_1 \in \mathbb{R}^{3\times 1}$$

$$H \underline{u}_i = \begin{bmatrix} \underline{h}_1^T \\ \underline{h}_2^T \\ \underline{h}_3^T \end{bmatrix} \underline{u}_i = \begin{bmatrix} \underline{h}_1^T \underline{u}_i \\ \underline{h}_2^T \underline{u}_i \\ \underline{h}_3^T \underline{u}_i \end{bmatrix}_{3\times 1}$$

$$\underline{h}_1^T \underline{u}_i \qquad \underline{u}_i \in \mathbb{R}^{3\times 1}$$

$$h_1 \in \mathbb{R}^{3\times 1}$$

$$h_1^T \in \mathbb{R}^{1\times 3}$$

$$\underline{h}_1^T \underline{u}_i = (\underline{h}_1^T \underline{u}_i)^T = \underline{u}_i^T \underline{h}_1$$

$$\underset{1\times 3}{h_1^T} \; \underset{3\times 1}{\underline{u}_i} \in \mathbb{R}^{1\times 1}$$

$$\underline{a}^T \underline{b} = \begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

$$\underline{b}^T \underline{a} = \begin{bmatrix} b_1 & b_2 & b_3 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = b_1 a_1 + b_2 a_2 + b_3 a_3$$

$$\underline{u}_i' \times H\underline{u}_i = 0$$

$$H\underline{u}_i = \begin{bmatrix} \underline{u}_i^T \underline{h}_1 \\ \underline{u}_i^T \underline{h}_2 \\ \underline{u}_i^T \underline{h}_3 \end{bmatrix}$$

$$A\underline{y} = b$$
$$A\underline{y} = 0$$

$$\underline{u}_i' \times H u_i = 0$$

$$\begin{bmatrix} 0 & -w_i' & y_i' \\ w_i' & 0 & -x_i' \\ -y_i' & x_i' & 0 \end{bmatrix}_{3\times 3} \begin{bmatrix} u_i^T h_1 \\ u_i^T h_2 \\ u_i^T h_3 \end{bmatrix}_{3\times 1} = 0$$

3 eqn

$$\begin{bmatrix} 0 - w_i' \underline{u}_i^T \underline{h}_1 + y_i' u_i^T h_3 \\ w_i' \underline{u}_i^T \underline{h}_1 + 0 - x_i' \underline{u}_i^T \underline{h}_2 \\ -y_i' u_i^T h_1 + x_i' u_i^T h_2 + 0 \end{bmatrix}_{3\times 1} = 0_{3\times 1}$$

$$\begin{array}{c} \\ 3 \end{array} \begin{array}{c} \overset{3}{\longleftrightarrow} \quad \overset{3}{\longleftrightarrow} \quad \overset{3}{\longleftrightarrow} \\ \underbrace{\left[\begin{array}{c|c|c} 0_{3\times 1}^T & -w_i' \underline{u}_i^T & y_i' u_i^T \\ w_i' u_i^T & 0_{3\times 1}^T & -x_i' u_i^T \\ -y_i' u_i^T & x_i' u_i^T & 0_{3\times 1}^T \end{array}\right]}_{A_{3\times 9}} \end{array} \underbrace{\begin{bmatrix} h_1 \\ h_2 \\ h_3 \end{bmatrix}_{9\times 1}}_{y} = 0_{3\times 1}$$

} 2 lin independents

$$A\underline{y} = 0$$

$$A_i \begin{bmatrix} 0_{3\times 1}^T & -w_i' \underline{u}_i^T & y_i' u_i^T \\ w_i' \;\; ^T & ^T & -x_i' u_i^T \end{bmatrix}_{2\times 9} \begin{bmatrix} h_1 \\ h_2 \\ h_3 \end{bmatrix}_{9\times 1} = 0_{2\times 1}$$

$$\begin{array}{l} \\ \text{2eqn} \\ \text{2eqn} \\ \text{2eqn} \\ \\ \end{array} \underbrace{\overset{\longleftarrow 9 \longrightarrow}{\begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{bmatrix}_{8\times 9}}}_{A} \underbrace{\begin{bmatrix} h_1 \\ h_2 \\ h_3 \end{bmatrix}_{9\times 1}}_{\underline{y}} = 0_{8\times 1}$$

$$\underline{y} = 0$$

$$A\underline{y} = 0$$

$$\underline{y} \in N(A)$$

$$\underset{8\times 9}{A} = \underset{8\times 8}{U}\,\Sigma\,\underset{9\times 9}{V^T} \qquad \text{SVD}$$

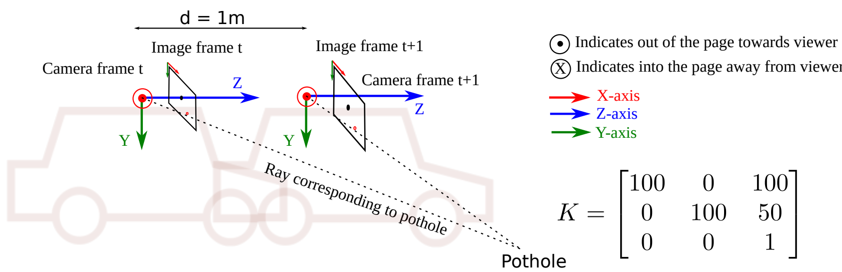
$$\text{rank}(A) = 8 \qquad 9 - 8 = 1$$

$$y \in N(A) = \underline{v}_9 \qquad V = \begin{bmatrix} \underline{v}_1 & \cdots & v_9 \end{bmatrix}$$

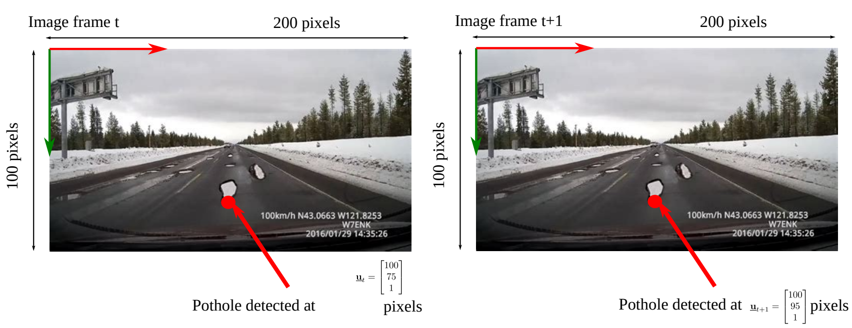
$$\begin{bmatrix} h_1 \\ h_2 \\ h_3 \end{bmatrix}_{9\times 1} = \underline{v}_{9} \qquad H = \begin{bmatrix} h_1^T \\ h_2^T \\ h_3^T \end{bmatrix}$$

Problem 1(b): *Find the 3D position of the pothole in the $t+1$ coordinate frame, in terms of $d = 1$ (the movement of the camera), image-coordinates of the pothole $\underline{\mathbf{u}}_t$, $\underline{\mathbf{u}}_{t+1}$ (provided in figure), camera matrix K (provided in figure). The car has moved from directly forward along Z_t-axis by $d = 1m$ without any rotation. We get two images at time t and at $t+1$. The detection of the pothole at time t is $\underline{\mathbf{u}}_t = [100, 75, 1]^\top$ and $\underline{\mathbf{u}}_{t+1} = [100, 95, 1]^\top$. Provide the formula or pseudo-code for computing the pothole coordinates. You may or may not need the following equations:*

1. *Projection of a 3D point $\mathbf{X}_c \in \mathbb{R}^3$ to an image point $\underline{\mathbf{u}} \in \mathbb{P}^2$ by pin hole camera model $\underline{\mathbf{u}} = K\mathbf{X}_c$, where K is the camera calibration matrix.*
2. *Transforming a coordinate from one coordinate frame to another $X_c = {}^cR_w X_c + {}^ct_w$ using given rotation matrix ${}^cR_w$ and translation ${}^ct_w$*
3. *The parameteric representation of a 3D line is $\mathbf{x} = \lambda\mathbf{d} + \mathbf{x}_0$, where $\mathbf{d} \in \mathbb{R}^3$ is the direction of the line, $\mathbf{x}_0 \in \mathbb{R}^3$ is any point on the line and $\lambda \in \mathbb{R}$ is a scalar.*



$$K = \begin{bmatrix} 100 & 0 & 100 \\ 0 & 100 & 50 \\ 0 & 0 & 1 \end{bmatrix}$$



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Problem 2 *Find the minimum point of the function,* $f(\mathbf{y}) = \mathbf{y}^\top A^\top A(\mathbf{y}-\mathbf{d}) - \mathbf{y}^\top \mathbf{b} + \mathbf{c}^\top \mathbf{y} + e$. *Let* $\mathbf{y} \in \mathbb{R}^{n\times 1}$ *be a n-dimensional vector and sizes of* $A, \mathbf{b}, \mathbf{c}, \mathbf{d}, e$ *be such that matrix multiplication and addition is valid. Also assume that* $A^\top A$ *is full rank, hence invertible. You may or may not need the following equations,*

1. $\frac{\partial}{\partial \mathbf{y}} \mathbf{y}^\top Q \mathbf{y} = 2Q\mathbf{y}$.
2. $\frac{\partial}{\partial \mathbf{y}} \mathbf{q}^\top \mathbf{y} = \frac{\partial}{\partial \mathbf{y}} \mathbf{y}^\top \mathbf{q} = \mathbf{q}$.

(10 marks).

$$y^T A^T A y - \underbrace{y^T A^T A d}$$

$$\frac{\partial}{\partial y} y^T \underbrace{\boxed{A^T A d}}_{\substack{(n\times n)(n\times 1) \\ n\times 1}} = \, ?$$

$d^T A^T A$ — 1

$A^T A d$ — 2

$\underbrace{\underbrace{d}_{n\times 1} \underbrace{A^T A}_{n\times n}}$ — 3

Problem 3 *Find the 3D line in parameteric representation that is formed by the intersection of two planes* $\mathbf{p}^\top \underline{\mathbf{x}} = 0$ *(with* $\mathbf{p} = [1,2,3,4]^\top$*) and* $\mathbf{q}^\top \underline{\mathbf{x}} = 0$ *where* $\mathbf{q} = [-3,2,1,4]^\top$*. You may or may not need the following equations,*

1. *Pseudo-inverse of a tall matrix* $A^\dagger = (A^\top A)^{-1}A^\top$.
2. *Pseudo-inverse of a fat matrix* $A^\dagger = A^\top(AA^\top)^{-1}$.

(10 marks)

$$\underline{p}^T \underline{x} = 0 \qquad \underline{p} \in \mathbb{R}^{4\times 1} \qquad \underline{x} \in \mathbb{R}^{4\times 1}$$

$$\underline{q}^T \underline{x} = 0 \qquad \underline{q} \in \mathbb{R}^{4\times 1}$$

$$\underbrace{\begin{bmatrix} \underline{p}^T \\ \underline{q}^T \end{bmatrix}_{2\times 4}}_{A} \underline{x}_{4\times 1} = 0_{2\times 1}$$

$$A\underline{x} = 0$$

$$\underline{x} = \mathcal{N}(A)$$

$$A_{2\times 4} = U_{2\times 2} \Sigma V^T_{4\times 4} \qquad \text{SVD}$$

$$\text{rank}(A) = 2$$

$$V = [\underline{v}_1, \underline{v}_2, \underline{v}_3, \underline{v}_4]_{4\times 4}$$

$$\underline{x} = \mathcal{N}(A) = \lambda_3 \underline{v}_3 + \lambda_4 \underline{v}_4 \qquad \underline{v}_3, \underline{v}_4 \in \mathbb{R}^{4\times 1}$$

$$\underline{x} = \lambda_3 \begin{bmatrix} v_{31} \\ v_{32} \\ v_{33} \\ v_{34} \end{bmatrix} + \lambda_4 \begin{bmatrix} v_{41} \\ v_{42} \\ v_{43} \\ v_{44} \end{bmatrix} \qquad \mathbb{P}^3$$

$$\boxed{\underline{x} = \underline{v}_3 + \frac{\lambda_4}{\lambda_3}\underline{v}_4}$$

$$\underline{x} = \lambda \underline{d} + \underline{x}_0$$

$$\underline{x} = \underline{v}_3 + t\,\underline{v}_4$$

$$\boxed{\vec{x} = \frac{1}{v_{34} + t v_{44}}\left[\underline{v}_3\right]_{1:3} + \frac{t}{v_{34} + t v_{44}}\left[v_4\right]_{1:3}}$$

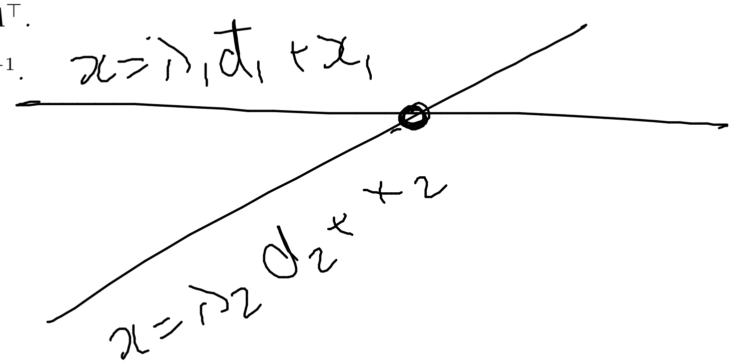
Problem 4 *Find the point on the intersection of following 3D lines* $\mathbf{x} = \lambda_1 \mathbf{d}_1 + \mathbf{x}_1$ *and* $\mathbf{x} = \lambda_2 \mathbf{d}_2 + \mathbf{x}_2$. *Here* $\lambda_1 \in \mathbb{R}$ *and* $\lambda_2 \in \mathbb{R}$ *are the free parameters. The rest of the parameters have the following values*

$$\mathbf{d}_1 = [1, 2, 0]^\top, \mathbf{d}_2 = [-2, 1, 0]^\top, \mathbf{x}_1 = [1, 2, 0]^\top, \mathbf{x}_2 = [4, 5, 0]^\top.$$

You may or may not need the following equations,

1. *Pseudo-inverse of a tall matrix* $A^\dagger = (A^\top A)^{-1} A^\top$.
2. *Pseudo-inverse of a fat matrix* $A^\dagger = A^\top (A A^\top)^{-1}$.

(10 marks).



$$x = \lambda_1 \underline{d}_1 + \underline{x}_1 = \lambda_2 \underline{d}_2 + \underline{x}_2$$

$$\lambda_1 \underline{d}_1 - \lambda_2 \underline{d}_2 = \underline{x}_2 - \underline{x}_1$$

$$\begin{bmatrix} \underline{d}_1 & | & -\underline{d}_2 \end{bmatrix}_{3\times 2} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = x_2 - x_1 \qquad A\underline{y} = b$$

$$\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \underbrace{\begin{bmatrix} d_1 & -d_2 \end{bmatrix}}_{A}{}^\dagger (x_2 - x_1)$$

$$(A^\top A)^{-1} A^\top (x_2 - x_2)$$

Problem 5 *Find the point of intersection of the 3D line* $\mathbf{x} = \lambda\mathbf{d} + \mathbf{x}_0$ *with the 3D plane* $\mathbf{p}^\top \underline{\mathbf{x}} = 0$. *The parameters have the following values*

$$\mathbf{d} = [1, 2, 0]^\top, \mathbf{x}_0 = [3, 4, 5]^\top, \mathbf{p} = [1, 2, 0, 7]^\top.$$

(10 marks).