ECE 417 Midterm 1

Vikas Dhiman

Feb 18th, 2021

(1) Student name: Student email:

About the exam

1. There are total 5 problems. You must attempt all 5.
2. Maximum marks: 50 (70 with bonus).
3. Maximum time allotted: 50 min
4. Calculators are allowed.
5. One US Letter size or A4 size cheat sheet (both-sides) is allowed.

Problem 1 *What are the two criteria for a* 3×3 *matrix* A *to be a valid rotation matrix? (5 min, 5 marks)*

$A^{-1} A = I$

$A^{-1} = A^T$

$$\left.\begin{array}{l} A A^T = I \\ A^T A = I \end{array}\right\} \text{orthogonality}$$

$\det(A) = 1$

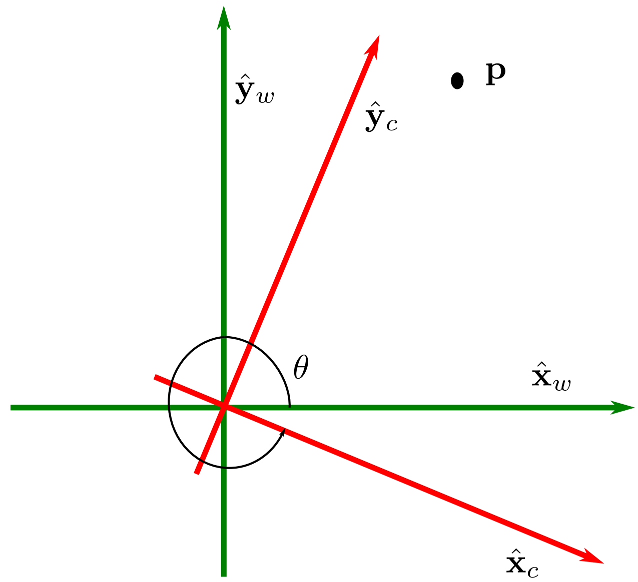


Figure 1: Rotation between red frame and green coordinate frame.

Problem 2 *For Fig 1, write the rotation matrix* $R(\theta)$ *that converts between coordinates of point from red coordinate frame* $\mathbf{p}_c$ *to green coordinate frame* $\mathbf{p}_w$ *such that* $\mathbf{p}_w = R(\theta)\mathbf{p}_c$*. (Optional part) Test your formula if it looks correct for* $\theta = -30°$, $\mathbf{p}_c = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$*. (5 min, 5 marks)*

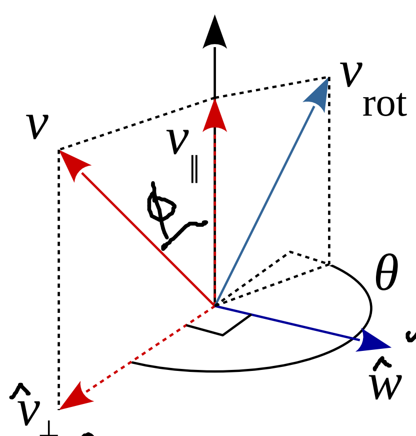
✓ $$R(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

$$\|\underline{a} \times \underline{b}\| = \|a\|\|b\| \sin \phi$$

$$\hat{v}_\perp = \underline{\hat{w}} \times \underline{\hat{k}} = \frac{(k \times v) \times \hat{k}}{\|k \times v\|}$$

$$\hat{w} = \hat{k} \times \frac{v}{\|v\|}$$

$$\boxed{\hat{w} \neq \hat{k} \times \frac{v}{\|v\|}}$$



ϕ

$\hat{v}_\perp = ?$

$$\hat{w} = ? = \frac{k \times v}{\|k \times v\|}$$

Figure 2: Rotation of point $\mathbf{v}$ around axis $\mathbf{k}$ by angle θ

Problem 3 *In Figure 2, we are rotating point* $\mathbf{v}$ *around axis unit-vector* $\hat{\mathbf{k}}$ *by an angle* θ*.* $\mathbf{v}_\perp$ *lies in the plane of* $\mathbf{v}$ *and* $\hat{\mathbf{k}}$ *and is orthogonal (perpendicular) to* $\mathbf{k}$*.* $\mathbf{w}$ *is perpendicular to the plane of* $\mathbf{v}$ *and* $\hat{\mathbf{k}}$*. First, write the unit-vector* $\hat{\mathbf{w}}$ *in terms of* $\mathbf{v}$ *and* $\hat{\mathbf{k}}$*. Then write unit-vector* $\hat{\mathbf{v}}_\perp$ *in terms of* $\mathbf{v}$ *and* $\hat{\mathbf{k}}$*. (10 min, 10 marks)*

Problem 4 *Using the fact that at minima (or maxima) of a differentiable function $f(\mathbf{x})$, $\nabla_{\mathbf{x}} f(\mathbf{x}) = 0$, find the minima of the following function,*

$$f(\mathbf{x}) = \mathbf{x}^\top A^\top A\mathbf{x} + 2\mathbf{b}^\top \mathbf{x} + c \tag{1}$$

(10 min, 10 marks)

$x \in \mathbb{R}^n$ $\quad b \in \mathbb{R}^n$

$A \in \mathbb{R}^{m\times n}$ $\quad c \in \mathbb{R}$

$$\min_x f(x)$$

$$\nabla_x f(x) = \frac{\partial}{\partial x} f(x) = 0$$

$$\frac{\partial}{\partial x}(x^T Q x) = 2Qx$$

$$\frac{\partial}{\partial x} a^T x = a$$

$$\frac{\partial}{\partial x} f(x) = \begin{bmatrix} \frac{\partial f(x)}{\partial x_1} \\ \vdots \\ \frac{\partial f(x)}{\partial x_n} \end{bmatrix}$$

$$\frac{\partial}{\partial x}\left[x^T A^T A x + 2b^T x + c\right] = 0$$

$$2A^T A x + 2b + 0 = 0$$

$$\Rightarrow (A^T A)x = -b$$

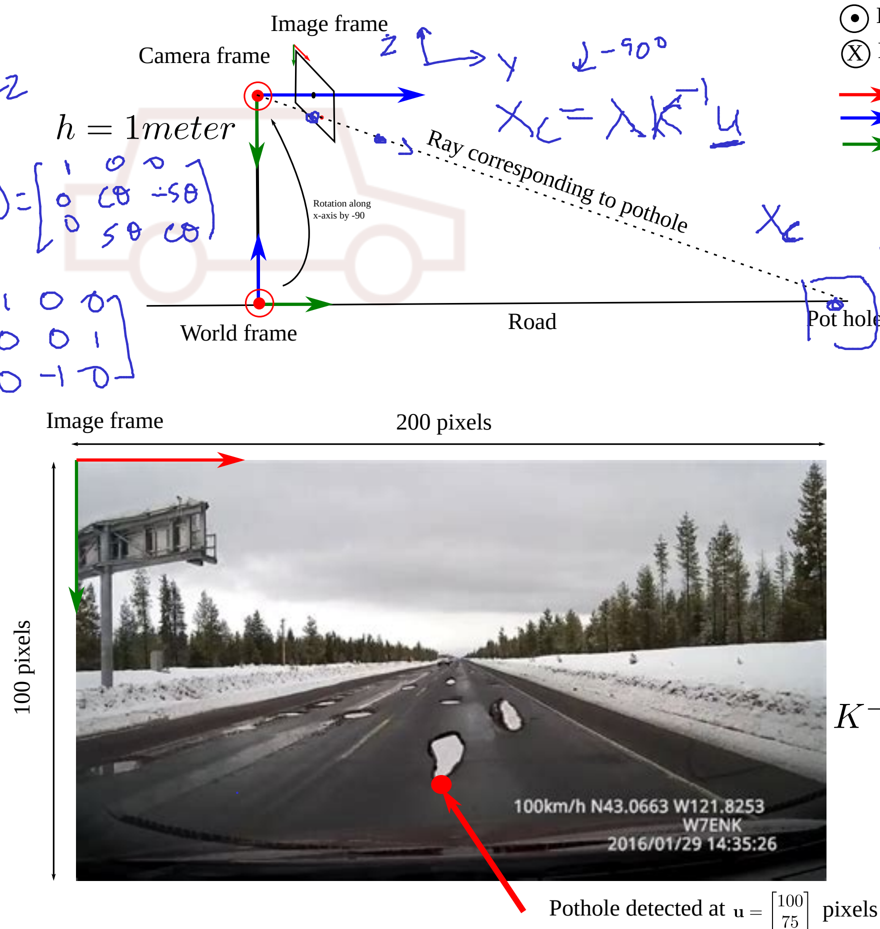
$$\Rightarrow x = -(A^T A)^{-1} b \quad \text{if } \det(A^T A) \neq 0$$

$$f(x) = (x - \mu_1)^T (A^T A)(x - \mu_1) + (x - \mu_2)^T (B^T B)(x - \mu_2)$$

$$= (x^T - \mu_1^T)(A^T A)(x - \mu_1) + (x^T - \mu_2^T) B^T B (x - \mu_2)$$

$$= x^T$$

⊙ Indicates out of the page towards viewer
ⓧ Indicates into the page away from viewer
X-axis
Z-axis
Y-axis



y-z

$R(-90) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\theta & -s\theta \\ 0 & s\theta & c\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$

$x_c = \lambda K^{-1} \underline{u}$

x_c

$$\mathbf{X}_w = \begin{bmatrix} ? \\ ? \\ 0 \end{bmatrix}$$

$x_c \Rightarrow \lambda K^{-1} \underline{u}$

$x_I = R(\theta) x_c$

$x_w = R(\theta) x_c + \underline{t}$

$t = \begin{bmatrix} 0 \\ 0 \\ h \end{bmatrix}$

$$K = \begin{bmatrix} 100 & 0 & 100 \\ 0 & 100 & 50 \\ 0 & 0 & 1 \end{bmatrix}$$

$\underline{x_w} = R(\theta)\lambda K^{-1}\underline{u} + \underline{t}$

$\lambda \{R(\theta) K^{-1} \underline{u}\}_3 + h = 0$

$$K^{-1} = \begin{bmatrix} 0.01 & 0 & -1 \\ 0 & 0.01 & -0.5 \\ 0 & 0 & 1 \end{bmatrix}$$

$\lambda = \dfrac{-h}{\{R(\theta) K^{-1} \underline{u}\}_3}$

$\lambda \underline{u} = K X_c$

$\underline{u} = \begin{bmatrix} 100 \\ 75 \\ 1 \end{bmatrix}$

Figure 3: Image road triangulation

Problem 5 *In figure 3 find the 3D position of pothole in the World coordinate frame, in terms of h and K. The Camera mounted directly on top of the world frame. The road is a perfect plane and the pothole lies on the road plane. You do not need to substitute in the values, providing a formula or pseudo-code for computing the pothole coordinates is enough. (20 min, 20 marks, Bonus marks: 20)*

Pinhole camera mod

① Img pt $\underline{u}$ → Ray (λ) $\lambda \in \mathbb{R}$
$x_c = \lambda K^{-1} \underline{u}$

R, t

② Ray (λ) → w → world frame
camera frame x_c → $x_w = R(\theta) \underline{x_c} + t$

Algebra

③ Ray(λ) intersection with Roadplane ($z_w = 0$)