ECE 417/598: Homework 3

Max marks: 120

Due on Feb 16th, 2021, midnight, 11:59 PM.

1 Linear Algebra review [1]

Problem 1 *More general than multiplication by columns is block multiplication. If matrices are separated into blocks (submatrices) and their shapes make block multiplication possible, then it is allowed:*

$$\left[\begin{array}{cc|c} x & x & x \\ x & x & x \\ \hline x & x & x \end{array}\right] \left[\begin{array}{cc|c} x & x & x \\ x & x & x \\ \hline x & x & x \end{array}\right] \tag{1}$$

$$\textit{or} \left[\begin{array}{cc|cc} x & x & x & x \\ x & x & x & x \end{array}\right] \left[\begin{array}{cc} x & x \\ x & x \\ \hline x & x \\ x & x \end{array}\right] \textit{or} \tag{2}$$

(a) Replace those x's by numbers and confirm that block multiplication succeeds. (ungraded, 30 min).

Problem 2 *Recall the following properties of determinant* $\det(.)$,

$$\det(AB) = \det(A)\det(B) \tag{3}$$

$$\det(\begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix}) = \det(\begin{bmatrix} A & 0 \\ 0 & I \end{bmatrix}\begin{bmatrix} I & 0 \\ 0 & D \end{bmatrix}) \tag{4}$$

$$= \det(A)\det(D) \tag{5}$$

Another property,

$$\det\begin{bmatrix} I & B \\ 0 & D \end{bmatrix} = \det(D) \tag{6}$$

Also note that by block-column elimination (col2 = col2 - col1 $A^{-1}B$ *),*

$$\begin{bmatrix} A & B \\ 0 & D \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix}\begin{bmatrix} I & A^{-1}B \\ 0 & I \end{bmatrix} \tag{7}$$

Using the above properties, prove that:

$$\det\begin{bmatrix} A & B \\ 0 & D \end{bmatrix} = \det(A)\det(D) \tag{8}$$

. (ungraded. 30 min).

Problem 3 *If* A *is* m *by* n *and* B *is* n *by* m*, show that*

$$\det\begin{bmatrix} 0_{m\times m} & A_{m\times n} \\ -B_{n\times m} & I_{n\times n} \end{bmatrix} = \det(AB) \tag{9}$$

Hint: Postmultiply LHS by $\det\begin{bmatrix} I_{m\times m} & 0_{m\times n} \\ B_{n\times m} & I_{n\times n} \end{bmatrix}$. *(ungraded. 30 min)*

Problem 4 *Define the following:*

1. *The column space (also called the range) of* A*, denoted by* $\mathcal{R}(A)$.
2. *The nullspace of* A*, denoted by* $\mathcal{N}(A)$.

Compute the rank, basis of column space and the basis of null space of a matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 2 & 4 & 6 \end{bmatrix}$. *(ungraded. 30 min).*

Problem 5 *Recall block elimination rules:*

$$\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \tag{10}$$

Block elimination gives (row 2 = row 2 - CA^{-1} *row 1), if the pivot block* A *is invertible,*

$$\begin{bmatrix} I & 0 \\ -CA^{-1} & I \end{bmatrix}\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A & B \\ 0 & D - CA^{-1}B \end{bmatrix} \tag{11}$$

The matrix $D - CA^{-1}B$ *is called a Schur's complement. Show that*

$$\det\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \det(A)\det(D - CA^{-1}B) \tag{12}$$

(ungraded. 30 min).

2 Older lectures

Problem 6 *Review: A 4×4 transformation matrix* ${}^{C}T_W \in \mathbb{R}^{4\times4}$ *that transforms coordinates from* W *to* C *by* $\underline{\mathbf{x}}_C = {}^{C}T_W \underline{\mathbf{x}}_W$ *is of the form:*

$$ {}^{C}T_W = \begin{bmatrix} {}^{C}R_W & {}^{C}\mathbf{t}_W \\ \mathbf{0}^\top & 1 \end{bmatrix}. $$

where $\mathbf{x}_W \in \mathbb{R}^3$ *and* $\mathbf{x}_C \in \mathbb{R}^3$ *are 3-D vectors representing the coordinates in* W *and* C *coordinate frame, respectively.* $\underline{\mathbf{x}} = [\mathbf{x}^\top, 1]^\top$ *denotes the homogeneous coordinate of* $\mathbf{x}$. ${}^{C}R_W \in \{R \in \mathbb{R}^{3\times3} | RR^\top = I \text{ and } \det(R) = 1\}$ *is the rotation matrix from* W *to* C *and* ${}^{C}\mathbf{t}_W \in \mathbb{R}^3$ *is the position of* W *origin in* C.

The opposite transformation from C *to* W *is simply the inverse of the matrix* ${}^{W}T_C = {}^{C}T_W^{-1}$.

1. *For any point* $\mathbf{x}_W$ *in the* W *coordinate, the corresponding coordinates in* C *coordinate is given by* $\mathbf{x}_C = {}^{C}R_W \mathbf{x}_W + {}^{C}\mathbf{t}_W$. *Prove that* $\mathbf{x}_W = {}^{C}R_W^\top \mathbf{x}_C - {}^{C}R_W^\top {}^{C}\mathbf{t}_W$.

2. *As a consequence of above proof write* ${}^{W}T_C$ *in the following form:*

$$ {}^{W}T_C = {}^{C}T_W^{-1} = \begin{bmatrix} {}^{C}R_W^\top & -{}^{C}R_W^\top {}^{C}\mathbf{t}_W \\ \mathbf{0}^\top & 1 \end{bmatrix}. $$

. (15 marks. 15 min).

3 Feb 2 and Feb 9th lecture

Problem 7 *Recall that given a camera matrix* K *and a 3D point* $\mathbf{X} = [X, Y, Z]$, *the point can be projected to a point on the image plane as* $\underline{\mathbf{u}} = [u, v, 1]$ *as*

$$ \lambda \underline{\mathbf{u}} = K\mathbf{X}, \quad (13) $$

where λ *is a scalar. As long as the last row of camera-intrinsic matrixk* K *is* $[0, 0, 1]$, *which it typically is, then* $\lambda = Z$. *Note that by rules of block-wise matrix multiplication, if you have a* n *points* $\mathbf{P} = [\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_n] \in \mathbb{R}^{3\times n}$ *matrix, then you can get the scaled image points* $\mathbf{U} = [\lambda_1\underline{\mathbf{u}}_1, \lambda_2\underline{\mathbf{u}}_2, \dots, \lambda_n\underline{\mathbf{u}}_n] \in \mathbb{R}^{3\times n}$ *by a single matrix multiplication,*

$$ \mathbf{U} = K\mathbf{P} \quad (14) $$

Write a ROS-node that ***subscribes*** *to a point cloud and publishes a depth image of size* 480×640. *After projection you should get the 2D coordinates of the point in the image* $[u, v, 1]^\top = \frac{1}{\lambda}K\mathbf{X}$. *A depth image* $d[i, j]$ *is an array that contains depth values. You should intialize all the depth values from* $i \in \{1, \dots, 480\}, j \in \{1, \dots, 640\}$ *as* $d[i, j] = 0$. *Since* u, v *will be real numbers, you can then populate* $d[round(u), round(v)] = Z$. *If the points project outside the image boundaries, then you can reject those points. Assume that the point cloud is already in the camera coordinate frame. Assume the camera intrinsic matrix to be* $K = \begin{bmatrix} 700 & 0 & 319.5 \\ 0 & 700 & 239.5 \\ 0 & 0 & 1 \end{bmatrix}$. *Use the ROS-node: point_cloud_to_depth_img.cpp for converting depth image to a point cloud as a starting template. (2 hour, 50 marks).*

4 Feb 7th lecture

Problem 8 *Recall that pseudo-inverse for a m-rank tall* $(n > m)$ *matrix* $A \in \mathbb{R}^{n\times m}$ *is given by* $A^\dagger = (A^\top A)^{-1}A^\top$ *and for a n-rank* $(m > n)$ *fat matrix is given by is given by* $A^\dagger = A^\top(AA^\top)^{-1}$ *. Consider the system* $A\mathbf{x} = \mathbf{b}$ *given by*

$$ \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 4 \\ 5 \end{bmatrix} \quad (15) $$

1. *Solve the system using all 4 equations, for a least-square solution. (Hint: Use Pseudo-inverse of a tall matrix).*

2. *Solve the system using only first 3 equations. (Hint: Use matrix inverse.)*

3. *Solve the system using only first 2 equations. (Hint: Use Pseudo-inverse of a fat matrix).*

You can use Matlab, Python, C++ or a calculator. Show your work. (30 min. 30 mark)

Problem 9 *In the class, we found the minima* *of* $\|A\mathbf{x}-\mathbf{b}\|_2^2$*. Here's a brief review:*

$$\min_{\mathbf{x}} \|A\mathbf{x}-\mathbf{b}\|_2^2 \tag{16}$$
$$= \min_{\mathbf{x}} (A\mathbf{x}-\mathbf{b})^\top (A\mathbf{x}-\mathbf{b}) \tag{17}$$
$$= \min_{\mathbf{x}} (\mathbf{x}^\top A^\top - \mathbf{b}^\top)(A\mathbf{x}-\mathbf{b}) \tag{18}$$
$$= \min_{\mathbf{x}} (\mathbf{x}^\top A^\top - \mathbf{b}^\top)(A\mathbf{x}-\mathbf{b}) \tag{19}$$
$$= \min_{\mathbf{x}} \mathbf{x}^\top A^\top A\mathbf{x} - \mathbf{b}^\top A\mathbf{x} - \mathbf{x}^\top A^\top \mathbf{b} + \mathbf{b}^\top \mathbf{b} \tag{20}$$

Recall that a minimum (or maximum) point of a differentiable function $f(\mathbf{x})$*,* $f'(\mathbf{x}) = 0$*. Let us define vector derivative as*

$$\frac{\partial f(\mathbf{x})}{\partial \mathbf{x}} = \nabla_{\mathbf{x}} f(\mathbf{x}) = \begin{bmatrix} \frac{\partial f(\mathbf{x})}{\partial x_1} \\ \frac{\partial f(\mathbf{x})}{\partial x_2} \\ \vdots \\ \frac{\partial f(\mathbf{x})}{\partial x_n} \end{bmatrix} \tag{21}$$

You can verfiy that

$$\frac{\partial}{\partial \mathbf{x}} \mathbf{x}^\top Q\mathbf{x} = 2Q\mathbf{x} \tag{22}$$
$$\frac{\partial}{\partial \mathbf{x}} \mathbf{b}^\top \mathbf{x} = \mathbf{b} \tag{23}$$

At a minimum point $\mathbf{x}$*,*

$$\frac{\partial}{\partial \mathbf{x}} \mathbf{x}^\top A^\top A\mathbf{x} - \mathbf{b}^\top A\mathbf{x} - \mathbf{x}^\top A^\top \mathbf{b} + \mathbf{b}^\top \mathbf{b} = 0 \tag{24}$$
$$\text{or } 2A^\top A\mathbf{x} - 2A^\top \mathbf{b} = 0 \tag{25}$$
$$\text{or } \mathbf{x} = \underbrace{(A^\top A)^{-1} A^\top}_{A^\dagger} \mathbf{b} \tag{26}$$

Using the same process, find the minima of the following function, assumming $\Sigma_1, \Sigma_2 \in \mathbb{R}^{n\times n}$ *being invertible and positive-definite; and* $\mathbf{x}, \boldsymbol{\mu}_1, \boldsymbol{\mu}_2 \in \mathbb{R}^{n\times 1}$*.*

$$f(\mathbf{x}) = (\mathbf{x}-\boldsymbol{\mu}_1)^\top \Sigma_1^{-1} (\mathbf{x}-\boldsymbol{\mu}_1) + (\mathbf{x}-\boldsymbol{\mu}_2)^\top \Sigma_2^{-1} (\mathbf{x}-\boldsymbol{\mu}_2) \tag{27}$$

(15 marks. 15 min)

References

[1] Gilbert Strang. *Linear algebra and its applications.* Belmont, CA: Thomson, Brooks/Cole, 1988.