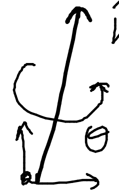
$\|\hat{k}\| = 1$

$\theta \in [0, \pi]$



Axis angle representation

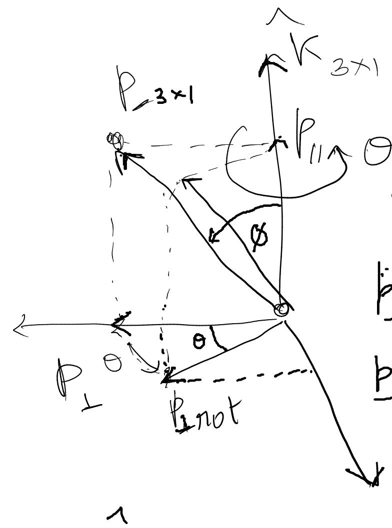
4 param

✓ $(\theta, \hat{k}) \longrightarrow R(\theta, \hat{k})$

3-param

$k = \theta \hat{k}$ $\quad \|k\| \le \pi$

$\lim_{\theta \to 0} k = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$



$p = p_{\parallel} + p_\perp$

$p_{\perp rot} = ?$

$p_{rot} = p_{\parallel} + p_{\perp rot}$

$(\theta, \hat{k}) \longrightarrow R(\theta, \hat{k})$

Rodrigues formula

$\hat{\omega} = \hat{k} \times \hat{p}$ $\quad | \quad$ $p_\perp = \omega \times \hat{k} = -\hat{k} \times \omega$

$\omega = \hat{k} \times p$ $\quad | \quad$ $= -\hat{k} \times (\hat{k} \times p)$

$\|p_\perp\| = |p| \sin\phi$

$\|\omega\| = |p| \sin\phi$

$\|p_\perp\| = \|\omega\|$

$p_{\perp rot} = \|p_\perp\| \cos\theta \, \hat{p}_\perp + \|p_\perp\| \sin\theta \, \hat{\omega}$

$= p_\perp \cos\theta + \omega \sin\theta$

$= -\hat{k} \times (\hat{k} \times p) \cos\theta + (\hat{k} \times p) \sin\theta$

$a \times b = \lfloor a \rfloor_\times b$ $\quad \lfloor a \rfloor_\times = \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix}_\times = \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix} \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix}$

$a \times b = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$

$= \hat{i}(a_y b_z - a_z b_y) - \hat{j}(a_x b_z - a_z b_x) + \hat{k}(a_x b_y - b_x a_y)$

$p_{\perp rot} = -\lfloor \hat{k} \rfloor_\times (\lfloor \hat{k} \rfloor_\times p) \cos\theta + \lfloor \hat{k} \rfloor_\times p \sin\theta$ $\quad \lfloor \hat{k} \rfloor_\times = K$

$= -K^2 p \cos\theta + K p \sin\theta$

$p = p_\parallel + p_\perp \Rightarrow p_\parallel = p - p_\perp = p - (-K^2 p) = p + K^2 p$

$p_{rot} = p_\parallel + p_{\perp rot} = p + K^2 p - K^2 p \cos\theta + K p \sin\theta$

$p_{rot} = p + K p \sin\theta + (1 - \cos\theta) K^2 p$

$= \left[I_{3\times 3} + K \sin\theta + (1 - \cos\theta) K^2 \right] p$

$$p_{rot} = R(\theta, \hat{k})\, \underline{p} \qquad R(\theta, \hat{k}) = \underset{3\times3}{I} + \underset{3\times3}{K} \sin\theta + (1-\cos\theta) K^2_{3\times3}$$

$$K = \begin{bmatrix} 0 & -k_z & k_y \\ k_z & 0 & -k_x \\ -k_y & k_x & 0 \end{bmatrix}$$

Rotation matrix $\longrightarrow$ axis - angle

$$R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} = \begin{bmatrix} I + K\sin\theta + (1-\cos\theta) K^2 \end{bmatrix}$$

$$\theta = \cos^{-1}\left(\frac{tr(R) - 1}{2}\right) \qquad tr(M) = \begin{bmatrix} M_{11} & & & \\ & M_{22} & & \\ & & \ddots & \\ & & & M_{nn} \end{bmatrix}$$

$$tr(ABC) = tr(BCA) = tr(CAB) = \sum_{i=1}^{n} M_{ii}$$

vv^T $v \in \mathbb{R}^{n\times1}$ $\mathbb{R}^{n\times n}$

$$tr(\underline{v}\,\underline{v}^T) = tr(v^T v)$$

$$\hat{k} = \begin{bmatrix} k_x \\ k_y \\ k_z \end{bmatrix} = \begin{bmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{bmatrix} \Big/ (2\sin\theta) \qquad \theta \neq 0, \pi$$

$\theta = 0$

$\Rightarrow \hat{k} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \theta = 0$

$$\boxed{R(\theta, \hat{k})\, \underline{v} = \underline{v}}$$

v = eigen vector corres. to $\lambda = 1$

$\theta = \pi$ $\qquad R = I + K^2$

$$K_x = \pm\sqrt{(r_{11}+1)/2}$$
$$K_y = \pm\sqrt{(r_{22}+1)/2}$$
$$K_z = \pm\sqrt{(r_{33}+1)/2}$$

Eigen vectors / values of a matrix M $\quad M \in \mathbb{R}^{n\times n}$

solutions $\quad M\underline{v} = \lambda \underline{v}$

$\lambda_1 \cdots \lambda_n$ $\quad$ Eigen values $\in \mathbb{R}$

$\underline{v}_1 \cdots \underline{v}_n$ $\quad$ " vectors $\in \mathbb{R}^{n\times1}$

$$M = \sum_i \lambda_i \underline{v}_i \underline{v}_i^T$$