Midterm 1 Review

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February 25, 2022

Contents

1 Linear algebra review

Definition 1 (Matrix)**.** *A real matrix A with n rows and m columns is defined as a set of real numbers $\{a_{11}, a_{12}, \dots, a_{nm}\}$, arranged in an 2D grid with n rows and m columns :*

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ a_{21} & a_{22} & \dots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nm} \end{bmatrix} \tag{1}$$

The set of all possible real matrices with n rows and m columns is denoted as $\mathbb{R}^{n\times m}$, where $\mathbb{R}$ denotes the set of all real numbers.

Any matrix A with with n rows and m columns is said to lie in the set of $\mathbb{R}^{n\times m}$. $A \in \mathbb{R}^{n\times m}$ is read aloud as "A lies in the set of all n cross m real matrices".

Definition 2 (Vector or Column vector)**.** *A column vector or a vector $\mathbf{x}$ is a matrix with only one column.*

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \tag{2}$$

The set of all possible real vectors with n rows is denoted as $\mathbb{R}^{n\times 1}$ or more simply $\mathbb{R}^n$.

A vector is denoted by bold-font small letter, for example, $\mathbf{x}, \mathbf{y}, \mathbf{z}$. A matrix is denoted by capital letters, A, B, M, P, K.

A matrix $A \in \mathbb{R}^{n\times m}$ is often denoted a set m column vectors of dimension $n \times 1$,

$$A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \dots & \mathbf{a}_m \end{bmatrix},$$
$$\text{where } \mathbf{a}_i = \begin{bmatrix} a_{i1} \\ a_{i2} \\ \vdots \\ a_{in} \end{bmatrix}, \qquad \text{for all } i \in \{1, \dots, m\}. \quad (3)$$

A block matrix is a matrix denoted in terms of other matrices,

$$A = \left[\begin{array}{ccc|ccc} b_{11} & \dots & b_{1q} & c_{11} & \dots & c_{1r} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ b_{p1} & \dots & b_{pq} & c_{1s} & \dots & c_{sr} \\ \hline e_{11} & \dots & e_{1v} & d_{11} & \dots & d_{1x} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ e_{u1} & \dots & e_{uv} & d_{1w} & \dots & d_{wx} \end{array}\right] \quad (4)$$
$$= \begin{bmatrix} B & C \\ E & D \end{bmatrix}, \text{ where } B, C, E, D \text{ are matrices.} \quad (5)$$

Definition 3 (Square matrix)**.** *A matrix is said to be square if its number of columns is same as the number of rows. That is matrix $A \in \mathbb{R}^{n\times m}$ is said to be square matrix if $m = n$.*

Definition 4 (Diagonal of a square matrix)**.** *Let A be a square matrix $A \in \mathbb{R}^{n\times n}$ with entries:*

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} \quad (6)$$

The diagonal of a square matrix A is defined to be the vector

$$diag(A) = \begin{bmatrix} a_{11} \\ a_{22} \\ \vdots \\ a_{nn} \end{bmatrix}$$

Definition 5 (Trace of a square matrix)**.** *Trace of a square matrix A is defined as the sum its diagonal elements,*

$$\operatorname{tr}(A) = \sum_{i=1}^{n} a_{ii}$$

.

Definition 6 (Identity matrix)**.** *An identity matrix I of size n is a square matrix with all its diagonal entries as 1 and non-diagonal entries as 0.*

$$I = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix} \quad (7)$$

1.1 Matrix operations

1.1.1 Transpose

Definition 7 (Transpose)**.** *The matrix transpose $A^\top$ of a matrix A is defined as a matrix where rows of matrix A are the columns of $A^\top$ and vice-versa.*

$$A^\top = \begin{bmatrix} a_{11} & a_{21} & \dots & a_{1n} \\ a_{12} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1m} & a_{2m} & \dots & a_{nm} \end{bmatrix} \quad (8)$$

In the matrix as set of m column vectors notation, the transpose is written as m row vectors $\mathbf{a}_i^\top$,

$$A^\top = \begin{bmatrix} \mathbf{a}_1^\top \\ \mathbf{a}_2^\top \\ \vdots \\ \mathbf{a}_m^\top \end{bmatrix}, \quad \mathbf{a}_i^\top = \begin{bmatrix} a_{i1} & a_{i2} & \dots & a_{in} \end{bmatrix},$$
$$\text{for all } i \in \{1, \dots, n\}. \quad (9)$$

1. If A has $n$rows and m columns, then $A^\top$ has m rows and n columns. If $A \in \mathbb{R}^{n\times m}$, then $A^\top \in \mathbb{R}^{m\times n}$.
2. The transpose of a transpose is matrix itself. $(A^\top)^\top = A$.
3. The transpose of a block matrix is block-wise transpose of each matrix,

$$\begin{bmatrix} B & C \\ E & D \end{bmatrix}^\top = \begin{bmatrix} B^\top & E^\top \\ C^\top & D^\top \end{bmatrix}$$

Definition 8 (Row vector)**.** *A row vector is Y is matrix with only one row*

$$Y = \begin{bmatrix} y_1 & y_2 & \dots & y_n \end{bmatrix} \quad (10)$$

It is common to denote row vectors as tranpose of a column vector. For example, the matrix Y shown above is typically represented $\mathbf{y}^\top$, where $\mathbf{y}$ is a column vector.

$$Y = \mathbf{y}^\top \qquad \text{where } \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} \tag{11}$$

1.1.2 Vector dot product

Before we define general matrix multiplication, it is easier to define matrix multiplication between a row vector and a column vector $\mathbf{x}^\top \in \mathbb{R}^{1\times n}$ and $\mathbf{y} \in \mathbb{R}^{n\times 1}$

$$\mathbf{x}^\top \mathbf{y} = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n = \sum_{i=1}^{n} x_i y_i \tag{12}$$

where $\mathbf{x}^\top = \begin{bmatrix} x_1 & \dots & x_n \end{bmatrix}$

and $\mathbf{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$.

Note that $\mathbf{x}^\top \mathbf{y}$ is same as the vector dot product or the vector inner-product,

$$\mathbf{x}^\top \mathbf{y} = \mathbf{x} \cdot \mathbf{y} = \|\mathbf{x}\| \|\mathbf{y}\| cos(\theta) = \mathbf{y}^\top \mathbf{x}, \tag{13}$$

where θ is the angle between vectors $\mathbf{x}$ and $\mathbf{y}$ and the vector norm or euclidean norm $\|.\|$ is defined as

$$\|\mathbf{x}\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2} = \sqrt{\mathbf{x}^\top \mathbf{x}} \tag{14}$$

Definition 9 (Unit vector)**.** *A unit vector, typically denoted with a hat,* $\hat{\mathbf{x}}$ *is a vector with euclidean norm as 1. That is* $\|\hat{\mathbf{x}}\| = 1$ *or equivalently* $\mathbf{x}^\top \mathbf{x} = 1$.

Definition 10 (Orthogonal vectors)**.** *Two vectors,* $\mathbf{x} \in \mathbb{R}^n$ *and* $\mathbf{y} \in \mathbb{R}^n$ *are said to be orthogonal if and only if their dot product is zero* $\mathbf{x}^\top \mathbf{y} = 0$.

Definition 11 (Orthonormal vectors)**.** *A set of vectors,* $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n \in \mathbb{R}^n$ *are said to be orthonormal if and only if they are all unit vectors* $\mathbf{x}_i^\top \mathbf{x}_i = 1$ *and they are pair-wise orthogonal,* $\mathbf{x}_i^\top \mathbf{x}_j = 0$ *for all* $i \neq j$.

1.1.3 Matrix multiplication

The matrix multiplication between matrix $A \in \mathbb{R}^{n\times m}$ and matrix $B \in \mathbb{R}^{m\times p}$ (note that A has m columns while B has m rows; the only case when matrix multiplication is defined) is easier defined if matrix A is written in terms of row vectors while matrix B is written in terms of column vectors. Let the matrix A is written in terms of row vectors $\mathbf{a}_i^\top \in \mathbb{R}^{1\times m}$ and the matrix B is written in terms of column vectors $\mathbf{b}_i \in \mathbb{R}^{m\times 1}$. Then the matrix multiplication $AB \in \mathbb{R}^{n\times p}$ is defined as the matrix,

$$AB = \begin{bmatrix} \mathbf{a}_1^\top \\ \mathbf{a}_2^\top \\ \vdots \\ \mathbf{a}_n^\top \end{bmatrix} \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \dots & \mathbf{b}_p \end{bmatrix} \tag{15}$$

$$= \begin{bmatrix} \mathbf{a}_1^\top \mathbf{b}_1 & \mathbf{a}_1^\top \mathbf{b}_2 & \dots & \mathbf{a}_n^\top \mathbf{b}_p \\ \mathbf{a}_2^\top \mathbf{b}_1 & \mathbf{a}_2^\top \mathbf{b}_2 & \dots & \mathbf{a}_n^\top \mathbf{b}_p \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{a}_n^\top \mathbf{b}_1 & \mathbf{a}_n^\top \mathbf{b}_2 & \dots & \mathbf{a}_n^\top \mathbf{b}_p \end{bmatrix} \tag{16}$$

Block matrix multiplication Block matrix multiplication works in a similar way as scalar multiplication as long as sub-matrix multiplication is properly defined,

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} P & Q \\ R & S \end{bmatrix} = \begin{bmatrix} AP + BR & AQ + BS \\ CP + DR & CQ + DS \end{bmatrix} \tag{17}$$

Definition 12 (Orthogonal matrices)**.** *A square matrix* A *is said to be orthogonal if and only if* $A^\top A = I$

1.1.4 Transpose of matrix multiplication

$$(AB)^\top = B^\top A^\top$$

1.1.5 Properties of trace operator

Trace is a linear operator:

$$\mathrm{tr}(\alpha A + \beta B) = \alpha\,\mathrm{tr}(A) + \beta\,\mathrm{tr}(B), \tag{18}$$

for compatible matrices A and B and scalars α and β.

2 Gaussian elimination or LDU factorization

A system of equations $A\mathbf{x} = \mathbf{b}$ can be solved by Gaussian elimination or equivalently factorizing A into triangular factorization also known as LDU factorization. After this factorization, the matrix $A \in \mathbb{R}^{n\times n}$ is factorized into lower-triangular matrix L, diagonal matrix D and uppper-triangular matrix U. Sometimes a permutation matrix P is also required. The factorization is such that $PA = LDU$. Let us write the matrix A in terms of its elements a_{ij}. Also let us denote the ith row of matrix A as $\mathbf{r}_i^\top = [a_{i1}, \dots, a_{in}]$.

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} = \begin{bmatrix} \mathbf{r}_1^\top \\ \mathbf{r}_2^\top \\ \vdots \\ \mathbf{r}_n^\top \end{bmatrix}. \tag{19}$$

To consider the simplest case, we assume that $a_{11} \neq 0$. If this is not the case, rearrange the rows of A such that $a_{11} \neq 0$. We want to make the first element of the second row to be zero. This can be achieved if we subtract $\mathbf{r}_2'^\top = \mathbf{r}_2^\top - \frac{a_{21}}{a_{11}}\mathbf{r}_1^\top$, where $\mathbf{r}_2'^\top$ is the new row after the operation. This is equivalent to multiplying both sides by a matrix:

$$\begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ -\frac{a_{21}}{a_{11}} & 1 & \dots & 0 & \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix} A$$
$$= \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ 0 & a_{22} - \frac{a_{21}}{a_{11}}a_{12} & a_{23} - \frac{a_{21}}{a_{11}}a_{13} & \dots & a_{2n} - \frac{a_{21}}{a_{11}}a_{1n} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{bmatrix}.$$

This makes sure that the first element of second row is zero. Since we want the represent matrix A as a factorization, we want to represent the matrix operation as one that restores the "un-does" the operation $\mathbf{r}_2'^\top = \mathbf{r}_2^\top - \frac{a_{21}}{a_{11}}\mathbf{r}_1^\top$, to get back the matrix A. Such an operation is $\mathbf{r}_2^\top = \mathbf{r}_2'^\top + \frac{a_{21}}{a_{11}}\mathbf{r}_1^\top$. So we can re-write the matrix A as,

$$A = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ \frac{a_{21}}{a_{11}} & 1 & \dots & 0 & \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix}$$
$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ 0 & a_{22} - \frac{a_{21}}{a_{11}}a_{12} & a_{23} - \frac{a_{21}}{a_{11}}a_{13} & \dots & a_{2n} - \frac{a_{21}}{a_{11}}a_{1n} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{bmatrix}.$$

A similar operation can be applied to all rows for $i \geq 2$ to make their first element to be zero, $\mathbf{r}_i'^\top = \mathbf{r}_i^\top - \frac{a_{i1}}{a_{11}}\mathbf{r}_1^\top$ for all $i \geq 2$.

$$A = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ \frac{a_{21}}{a_{11}} & 1 & \dots & 0 & \\ \frac{a_{31}}{a_{11}} & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{a_{n1}}{a_{11}} & 0 & 0 & \dots & 1 \end{bmatrix}$$
$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ 0 & a_{22} - \frac{a_{21}}{a_{11}}a_{12} & a_{23} - \frac{a_{21}}{a_{11}}a_{13} & \dots & a_{2n} - \frac{a_{21}}{a_{11}}a_{1n} \\ 0 & a_{32} - \frac{a_{31}}{a_{11}}a_{13} & a_{33} - \frac{a_{31}}{a_{11}}a_{13} & \dots & a_{3n} - \frac{a_{31}}{a_{11}}a_{1n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & a_{n2} - \frac{a_{n1}}{a_{11}}a_{12} & a_{n3} - \frac{a_{n1}}{a_{11}}a_{13} & \dots & a_{nn} - \frac{a_{n1}}{a_{11}}a_{1n} \end{bmatrix}.$$

Thus we have made first column zero below the diagonal on the second matrix. As long $a_{22}' = a_{22} - \frac{a_{21}}{a_{11}}a_{12} \neq 0$, a similar procedure can be applied to the make the second column zero below the diagonal. If $a_{22}' = 0$, then swap the second row with a row whos second element is non-zero. If all rows have second element as zero then this step is done, target to make the third column below to be zero. Denote the second column elements for row $i \geq 2$ by $a_{ij}' = a_{ij} - \frac{a_{i1}}{a_{11}}a_{1j}$. Then the operation $\mathbf{r}_i''^\top = \mathbf{r}_i'^\top - \frac{a_{i2}'}{a_{22}'}\mathbf{r}_2'^\top$ for all $i \geq 3$, will make the second column below diagonal to be zero. And the matrix factorization will look like,

$$A = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ \frac{a_{21}}{a_{11}} & 1 & \dots & 0 \\ \frac{a_{31}}{a_{11}} & \frac{a'_{32}}{a'_{22}} & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{a_{n1}}{a_{11}} & \frac{a'_{n2}}{a'_{22}} & 0 & \dots & 1 \end{bmatrix}$$
$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ 0 & a_{22} - \frac{a_{21}}{a_{11}} a_{12} & a_{23} - \frac{a_{21}}{a_{11}} a_{13} & \dots & a_{2n} - \frac{a_{21}}{a_{11}} a_{1n} \\ 0 & 0 & a'_{33} - \frac{a'_{32}}{a'_{22}} a'_{23} & \dots & a'_{3n} - \frac{a'_{32}}{a'_{22}} a'_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & a_{n3} - \frac{a'_{n1}}{a'_{22}} a'_{23} & \dots & a'_{nn} - \frac{a'_{n2}}{a'_{22}} a'_{2n} \end{bmatrix}.$$

One can continue this procedure until the second matrix is all 0's below the diagonal (upper-triangular) while maintaining that the first matrix is all 0's above the diagonal, (lower-triangular),

$$A = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ \frac{a_{21}}{a_{11}} & 1 & \dots & 0 \\ \frac{a_{31}}{a_{11}} & \frac{a'_{32}}{a'_{22}} & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{a_{n1}}{a_{11}} & \frac{a'_{n2}}{a'_{22}} & \frac{a''_{n3}}{a''_{n3}} & \dots & 1 \end{bmatrix}$$
$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ 0 & a_{22} - \frac{a_{21}}{a_{11}} a_{12} & a_{23} - \frac{a_{21}}{a_{11}} a_{13} & \dots & a_{2n} - \frac{a_{21}}{a_{11}} a_{1n} \\ 0 & 0 & a'_{33} - \frac{a'_{32}}{a'_{22}} a'_{23} & \dots & a'_{3n} - \frac{a'_{32}}{a'_{22}} a'_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & u_{nn} \end{bmatrix}.$$

The first matrix is denoted as L while the second matrix is denoted as DU. The second matrix can be factorized into diagonal matrix D and upper-triangular matrix U so that U matrix has diagonal elements as 1,

$$A = LDU$$

$$\text{where } L = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ \frac{a_{21}}{a_{11}} & 1 & \dots & 0 \\ \frac{a_{31}}{a_{11}} & \frac{a'_{32}}{a'_{22}} & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{a_{n1}}{a_{11}} & \frac{a'_{n2}}{a'_{22}} & \frac{a''_{n3}}{a''_{n3}} & \dots & 1 \end{bmatrix}$$

$$DU = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ 0 & a_{22} - \frac{a_{21}}{a_{11}} a_{12} & a_{23} - \frac{a_{21}}{a_{11}} a_{13} & \dots & a_{2n} - \frac{a_{21}}{a_{11}} a_{1n} \\ 0 & 0 & a'_{33} - \frac{a'_{32}}{a'_{22}} a'_{23} & \dots & a'_{3n} - \frac{a'_{32}}{a'_{22}} a'_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & u'_{nn} \end{bmatrix}$$

Let the elements of upper-triangular DU matrix be denoted by u'_{ij},

$$DU = \begin{bmatrix} u'_{11} & u'_{12} & \dots & u'_{1n} \\ 0 & u'_{22} & \dots & u'_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & u'_{nn} \end{bmatrix}.$$

Such a matrix can be factorized into diagonal matrix D and U where D has the diagonal elements of U while U rows are divided by the diagonal element,

$$D = \begin{bmatrix} u'_{11} & 0 & 0 & \dots & 0 \\ 0 & u'_{22} & 0 & \dots & 0 \\ 0 & 0 & u'_{33} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & u'_{nn} \end{bmatrix}.$$
$$U = \begin{bmatrix} 1 & \frac{u'_{12}}{u'_{11}} & \frac{u'_{13}}{u'_{11}} & \dots & \frac{u'_{1n}}{u'_{11}} \\ 0 & 1 & \frac{u'_{23}}{u'_{22}} & \dots & \frac{u'_{2n}}{u'_{22}} \\ 0 & 0 & 1 & \dots & \frac{u'_{3n}}{u'_{33}} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix}$$

Example 1. *Let A be the matrix,*

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 2 & 4 & 6 \end{bmatrix}$$

1. *Step 1: Row 2 = Row 2 - 4 times Row 1 or* $\mathbf{r}_2^{\prime\top} = \mathbf{r}_2^\top - 4\mathbf{r}_1^\top$.

$$A = \begin{bmatrix}1 & 2 & 3\\ 4 & 5 & 6\\ 2 & 4 & 6\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\ 4 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 2 & 3\\ 0 & -3 & -6\\ 2 & 4 & 6\end{bmatrix}$$

The first matrix "un-does" $\mathbf{r}_2^{\prime\top} = \mathbf{r}_2^\top - 4\mathbf{r}_1^\top$, *because it operates* $\mathbf{r}_2^\top = \mathbf{r}_2^{\prime\top} + 4\mathbf{r}_1^\top$ *on the second matrix to get back* A.

2. *Step 2: Row 3 = Row 3 - 2 times row 1 or* $\mathbf{r}_3^{\prime\top} = \mathbf{r}_3^\top - 2\mathbf{r}_1^\top$,

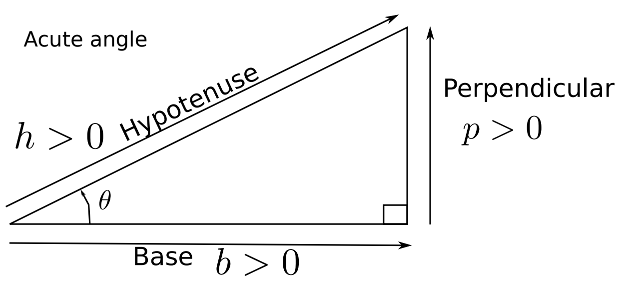
$$A = \begin{bmatrix}1 & 2 & 3\\ 4 & 5 & 6\\ 2 & 4 & 6\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\ 4 & 1 & 0\\ 2 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 2 & 3\\ 0 & -3 & -6\\ 0 & 0 & 0\end{bmatrix}$$

We do not need to do any more operations because we already got lower-triangular and upper-triangular decomposition. We can separate the diagonal from the upper-triangular matrix,

3. *Step 3: Separate the diagonal components from upper-triangular matrix,*

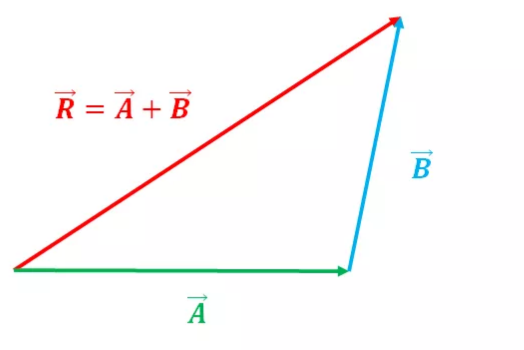
$$A = \underbrace{\begin{bmatrix}1 & 0 & 0\\ 4 & 1 & 0\\ 2 & 0 & 1\end{bmatrix}}_{L}\underbrace{\begin{bmatrix}1 & 0 & 0\\ 0 & -3 & 0\\ 0 & 0 & 0\end{bmatrix}}_{D}\underbrace{\begin{bmatrix}1 & 2 & 3\\ 0 & 1 & 2\\ 0 & 0 & 0\end{bmatrix}}_{U}$$

3 Trignometry review



$$\tan(\theta) = \frac{p}{b} \qquad \sin(\theta) = \frac{p}{h} \qquad \cos(\theta) = \frac{b}{h}$$

4 Triangle law of vector addition



5 2D Rotation matrix

Definition 13 (2D Cartesian Coordinate frame)**.** *A 2D cartesian coordinate frame is defined as a set of mutually orthogonal unit vectors* $\hat{\mathbf{x}} \in \mathbb{R}^2$ *and* $\hat{\mathbf{y}} \in \mathbb{R}^2$ *called the basis vectors* $B = [\hat{\mathbf{x}}, \hat{\mathbf{y}}]$ *along with an origin* $\mathbf{o} \in \mathbb{R}^2$. *Thus the tuple* $(B, \mathbf{o})$ *form a coordinate frame. A coordinate frame is denoted by curly braces around it, for example,* $\{C\}$ *or* $\{W\}$.

Example 2 (2D Coordinate frame)**.** *The figure 1 contains two coordinate frames the one shown in red and the one shown in green. Both have the same origin, but different basis vectors. The* $\{W\}$ *coordinate frame shown in green has basis vectors* $B_w = [\hat{\mathbf{x}}_w, \hat{\mathbf{y}}_w]$. *The same notation is used for the* $\{C\}$ *coordinate frame* $B_c = [\hat{\mathbf{x}}_c, \hat{\mathbf{y}}_c]$. *Note that the basis vectors of* $\{C\}$ *coordinate frame can be expressed in terms of* $\{W\}$ *coordinate frame by triangle law of vector addition,*

$$\begin{aligned}\hat{\mathbf{x}}_c &= |\overrightarrow{OA}|\hat{\mathbf{x}}_w + |\overrightarrow{AB}|\hat{\mathbf{y}}_w \\ \hat{\mathbf{y}}_c &= -|\overrightarrow{PQ}|\hat{\mathbf{x}}_w + |\overrightarrow{OP}|\hat{\mathbf{y}}_w\end{aligned} \tag{20}$$

In the triangle ΔOAB (Fig 1),

$$\cos(\theta) = \frac{|\overrightarrow{OA}|}{|\overrightarrow{OB}|} = \frac{|\overrightarrow{OA}|}{\|\hat{\mathbf{x}}_c\|} = |\overrightarrow{OA}| \tag{21}$$

$$\sin(\theta) = \frac{|\overrightarrow{AB}|}{|\overrightarrow{OB}|} = \frac{|\overrightarrow{AB}|}{\|\hat{\mathbf{x}}_c\|} = |\overrightarrow{AB}| \tag{22}$$

Similarly in the right triangle ΔOPQ (Fig 1),

$$\cos(\theta) = \frac{|\overrightarrow{OP}|}{|\overrightarrow{OQ}|} = \frac{|\overrightarrow{OP}|}{\|\hat{\mathbf{y}}_c\|} = |\overrightarrow{OP}| \tag{23}$$

$$\sin(\theta) = \frac{|\overrightarrow{PQ}|}{|\overrightarrow{OQ}|} = \frac{|\overrightarrow{PQ}|}{\|\hat{\mathbf{y}}_c\|} = |\overrightarrow{PQ}| \tag{24}$$

Putting these values back in (20)*, we get,*

$$\begin{aligned} \hat{\mathbf{x}}_c &= cos(\theta)\hat{\mathbf{x}}_w + sin(\theta)\hat{\mathbf{y}}_w \\ \hat{\mathbf{y}}_c &= -sin(\theta)\hat{\mathbf{x}}_w + cos(\theta)\hat{\mathbf{y}}_w \end{aligned} \tag{25}$$

These equations can be written in matrix notation as,

$$\begin{aligned} \hat{\mathbf{x}}_c &= \begin{bmatrix}\hat{\mathbf{x}}_w & \hat{\mathbf{y}}_w\end{bmatrix}\begin{bmatrix}cos(\theta)\\ sin(\theta)\end{bmatrix} = B_w\begin{bmatrix}cos(\theta)\\ sin(\theta)\end{bmatrix} \\ \hat{\mathbf{y}}_c &= \begin{bmatrix}\hat{\mathbf{x}}_w & \hat{\mathbf{y}}_w\end{bmatrix}\begin{bmatrix}-sin(\theta)\\ cos(\theta)\end{bmatrix} = B_w\begin{bmatrix}-sin(\theta)\\ cos(\theta)\end{bmatrix} \end{aligned} \tag{26}$$

The full basis matrix of coordinate frame $\{C\}$ can be written as

$$\begin{aligned} B_c &= \begin{bmatrix}\hat{\mathbf{x}}_c & \hat{\mathbf{y}}_c\end{bmatrix} \\ &= \begin{bmatrix} B_w\begin{bmatrix}cos(\theta)\\ sin(\theta)\end{bmatrix} & B_w\begin{bmatrix}-sin(\theta)\\ cos(\theta)\end{bmatrix}\end{bmatrix} \\ &= B_w\begin{bmatrix}cos(\theta) & -sin(\theta)\\ sin(\theta) & cos(\theta)\end{bmatrix} \end{aligned} \tag{27}$$

Definition 14 (2D Coordinates of a point)**.** *The coordinate of a point $\mathbf{p}$ in a given coordinate frame $\{W\}$ with basis vectors $B_w = [\hat{\mathbf{x}}_w, \hat{\mathbf{y}}_w]$ and origin $\mathbf{o}_w = \begin{bmatrix}o_x\\ o_y\end{bmatrix}$ is defined as the vector $\mathbf{p}_w = \begin{bmatrix}p_{wx}\\ p_{wy}\end{bmatrix}$ such that,*

$$\begin{aligned} \mathbf{p} &= (p_{wx} + o_x)\hat{\mathbf{x}}_w + (p_{wy} + o_y)\hat{\mathbf{y}}_w \\ &= \begin{bmatrix}\hat{\mathbf{x}}_w & \hat{\mathbf{y}}_w\end{bmatrix}\left(\begin{bmatrix}p_{wx}\\ p_{wy}\end{bmatrix} + \begin{bmatrix}o_x\\ o_y\end{bmatrix}\right) \\ &= B_w(\mathbf{p}_w + \mathbf{o}_w) \end{aligned} \tag{28}$$

Example 3 (Fig 1)**.** *The point $\mathbf{p}$ can be represented in coordinate frames $\{W\}$ and $\{C\}$. Let the projection on the basis $B_c = [\hat{\mathbf{x}}_c, \hat{\mathbf{y}}_c]$ be $\mathbf{p}_c$, while that on $B_w = [\hat{\mathbf{x}}_w, \hat{\mathbf{y}}_w]$ be $\mathbf{p}_w$. Since both the coordinate frames have same origin, we assume $\mathbf{o}_w = \mathbf{o}_c = \begin{bmatrix}0\\ 0\end{bmatrix}$. We have*

$$\mathbf{p} = B_w\mathbf{p}_w = B_c\mathbf{p}_c \tag{29}$$

Theorem 1 (2D Rotation matrix)**.** *In a coordinate transformation as given in Fig 1, the coordinates in frame $\{C\}$, $\mathbf{p}_c$ can be converted into coordinates in frame $\{W\}$, $\mathbf{p}_w$ with the same origin by using a rotation matrix ${}^WR_C(\theta)$,*

$$\begin{aligned} \mathbf{p}_w &= {}^WR_C(\theta)\mathbf{p}_c \\ \textit{where } {}^WR_C(\theta) &= \begin{bmatrix}cos(\theta) & -sin(\theta)\\ sin(\theta) & cos(\theta)\end{bmatrix} \end{aligned} \tag{30}$$

Proof. First note that the basis matrix of any coordinate frame $\{W\}$ is orthogonal,

$$\begin{aligned} B_w^\top B_w &= \begin{bmatrix}\hat{\mathbf{x}} & \hat{\mathbf{y}}\end{bmatrix}^\top \begin{bmatrix}\hat{\mathbf{x}} & \hat{\mathbf{y}}\end{bmatrix} \\ &= \begin{bmatrix}\hat{\mathbf{x}}^\top\\ \hat{\mathbf{y}}^\top\end{bmatrix}\begin{bmatrix}\hat{\mathbf{x}} & \hat{\mathbf{y}}\end{bmatrix} \\ &= \begin{bmatrix}\hat{\mathbf{x}}^\top\hat{\mathbf{x}} & \hat{\mathbf{x}}^\top\hat{\mathbf{y}}\\ \hat{\mathbf{y}}^\top\hat{\mathbf{x}} & \hat{\mathbf{y}}^\top\hat{\mathbf{y}}\end{bmatrix} \\ &= \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} = I \end{aligned} \tag{31}$$

Left-multiply $B_w^\top$ to both sides of (29)

$$B_w^\top B_w\mathbf{p}_w = B_w^\top B_c\mathbf{p}_c \tag{32}$$

Replace $B_w^\top B_w = I$.

$$I\mathbf{p}_w = B_w^\top B_c\mathbf{p}_c \text{ or } \mathbf{p}_w = B_w^\top B_c\mathbf{p}_c \tag{33}$$

Substitute value of B_c from (27), to get

$$\mathbf{p}_w = B_w^\top\left(B_w\begin{bmatrix}\cos(\theta) & -\sin(\theta)\\ \sin(\theta) & \cos(\theta)\end{bmatrix}\right)\mathbf{p}_c. \tag{34}$$

Again use $B_w^\top B_w = I$ to get,

$$\mathbf{p}_w = \begin{bmatrix}\cos(\theta) & -\sin(\theta)\\ \sin(\theta) & \cos(\theta)\end{bmatrix}\mathbf{p}_c. \tag{35}$$

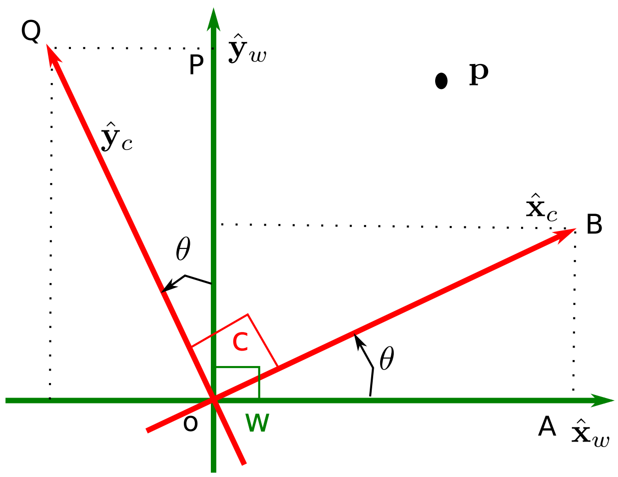


Figure 1: The coordinate frame $\{C\}$ is rotated around origin by an θ from coordinate frame $\{W\}$.

Defining ${}^{W}R_C(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$, we get the desired result.

□

Theorem 2 (Orthogonality of 2D Rotation matrices). *All 2D rotation matrices are orthogonal* $R^\top R = I$ *have determinant as one* $\det(R) = 1$*. If any square matrix* $A \in \mathbb{R}^{2\times 2}$ *is orthogonal* $A^\top A = I$ *and has determinant 1,* $\det(A) = 1$*, then it is a valid rotation matrix.*

Proof.

$$\begin{aligned} R^\top R &= \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}^\top \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \\ &= \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \\ &= \begin{bmatrix} \cos^2(\theta) + \sin^2(\theta) & -\cos(\theta)\sin(\theta) + \sin(\theta)\cos(\theta) \\ -\sin(\theta)\cos(\theta) + \cos(\theta)\sin(\theta) & \sin^2(\theta) + \cos^2(\theta) \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{aligned} \tag{36}$$

$$\begin{aligned} \det(R) &= \det \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \\ &= \cos^2(\theta) + \sin^2(\theta) = 1 \end{aligned} \tag{37}$$

Denote the columns of square matrix A which is orthogonal with determinant 1 as $A = [\mathbf{a}_1, \mathbf{a}_2]$. Since A is orthogonal, we have

$$\begin{aligned} A^\top A &= \begin{bmatrix} \mathbf{a}_1^\top \\ \mathbf{a}_2^\top \end{bmatrix} \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{a}_1^\top \mathbf{a}_1 & \mathbf{a}_1^\top \mathbf{a}_2 \\ \mathbf{a}_2^\top \mathbf{a}_1 & \mathbf{a}_2^\top \mathbf{a}_2 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \end{aligned} \tag{38}$$

This implies that $\mathbf{a}_1$ and $\mathbf{a}_2$ are mutually orthogonal unit vectors. Let $\mathbf{a}_1 = [\cos(\theta), \sin(\theta)]$ because any 2D unit vector can be written in cos,sin form, where $\theta = \text{arctan2}(a_{12}, a_{11})$. Next we know that $\mathbf{a}_1^\top \mathbf{a}_2 = 0$ and that $\mathbf{a}_2$ is unit vector. For any unit 2D vector $[u, v]^\top$, there are only two unit vectors perpendicular to it $[-v, u]^\top$ and $[v, -u]^\top$. Then we have only two options for $\mathbf{a}_2$ are either $[-\sin(\theta), \cos(\theta)]$ or $[\sin(\theta), -\cos(\theta)]$. But we also know that the determinant of A is 1. The second option for $\mathbf{a}_2$ leads to determinant of -1.

$$\det \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 \end{bmatrix} = \det \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ \sin(\theta) & -\cos(\theta) \end{bmatrix} = -1 \tag{39}$$

Hence, we have

$$A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 \end{bmatrix} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} = R(\theta)$$

□

6 2D Transformation matrix

To consider the rotation and translation case, we consider the case shown in Fig 2. We have an intermediate frame $\{I\}$ which has only rotation from $\{C\}$ frame. We assume that basis vectors $\{I\}$ are parallel to $\{W\}$ which make it translation only conversion. We can convert from $\mathbf{p}_c$ to $\mathbf{p}_I$ using the rotation matrix derived in the previous section,

$$\mathbf{p}_I = B_I^{-1} B_c \mathbf{p}_c = R(\theta)\mathbf{p}_c. \tag{40}$$

We can account for the translation of the frame $\mathbf{p}_I$ by noticing that the coordinate frames only differ in origin, such that $B_c \mathbf{o}_c = B_w(\mathbf{o}_w + {}^{w}\mathbf{t}_c)$, where the trans-

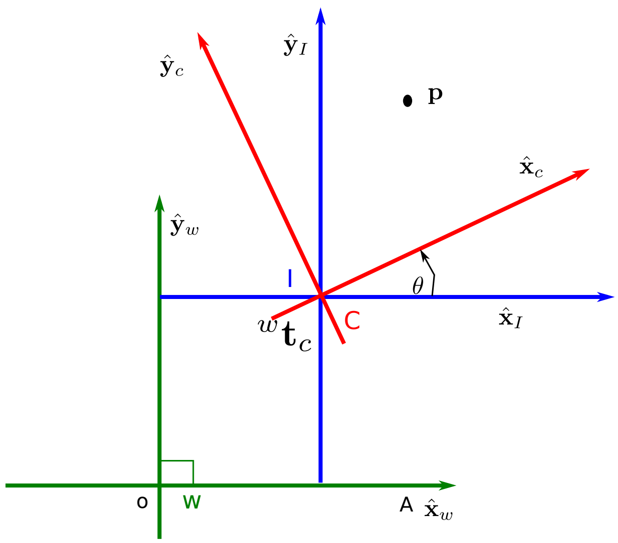


Figure 2: The coordinate frame $\{C\}$ is rotated around origin by an θ from coordinate frame $\{W\}$ and then shifted by translation ${}^{w}\mathbf{t}_c$.

lation ${}^{w}\mathbf{t}_c$ is measured in world coordinate frame.

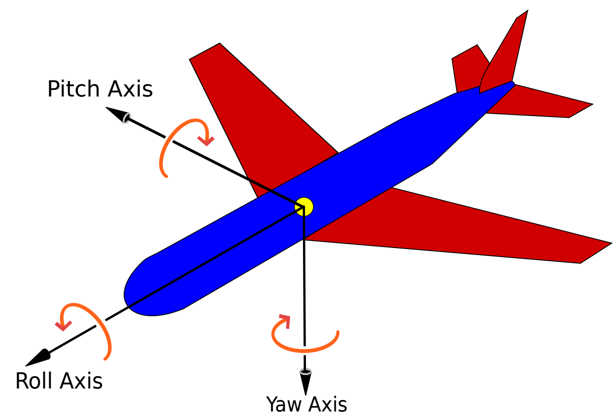
$$\begin{aligned}
&\mathbf{p} = B_c(\mathbf{p}_c + \mathbf{o}_c) = B_w(\mathbf{p}_w + \mathbf{o}_w)\\
\implies & B_c\mathbf{p}_c + B_c\mathbf{o}_c = B_w\mathbf{p}_w + B_w\mathbf{o}_w\\
\implies & B_c\mathbf{p}_c + (B_c\mathbf{o}_c - B_w\mathbf{o}_w) = B_w\mathbf{p}_w\\
\implies & B_c\mathbf{p}_c + B_w{}^{w}\mathbf{t}_c = B_w\mathbf{p}_w\\
\implies & B_w^\top B_c\mathbf{p}_c + {}^{w}\mathbf{t}_c = \mathbf{p}_w\\
\implies & \mathbf{p}_w = R(\theta)\mathbf{p}_c + {}^{w}\mathbf{t}_c
\end{aligned} \tag{41}$$

This relation is often written in terms of homogeneous coordinates which are obtained by appending 1 to euclidean coordinates $\underline{\mathbf{p}}_w = \begin{bmatrix}\mathbf{p}_w\\1\end{bmatrix}$ and $\underline{\mathbf{p}}_c = \begin{bmatrix}\mathbf{p}_c\\1\end{bmatrix}$.

The matrix that transforms homogeneous coordinates in one coordinate frame to another is called the transformation matrix. For 2D systems it is 3×3 matrix denoted by ${}^{w}T_c$,

$$\underline{\mathbf{p}}_w = \begin{bmatrix}R(\theta) & {}^{w}\mathbf{t}_c\\ \mathbf{0}^\top & 1\end{bmatrix}\underline{\mathbf{p}}_c = {}^{w}T_c\underline{\mathbf{p}}_c \tag{42}$$

7 Principal 3D Rotations



2D Rotation can be easily extended to rotation around an axis in 3D. Rotation around X-axis, Y-axis, Z-axis is respectively given by,

$$\begin{aligned}
R_x(\phi) = \text{Roll}(\phi) &= \begin{bmatrix}1 & 0 & 0\\ 0 & \cos(\phi) & -\sin(\phi)\\ 0 & \sin(\phi) & \cos(\phi)\end{bmatrix}\\
R_y(\theta) = \text{Pitch}(\theta) &= \begin{bmatrix}\cos(\phi) & 0 & \sin(\phi)\\ 0 & 1 & 0\\ -\sin(\phi) & & \cos(\phi)\end{bmatrix}\\
R_z(\psi) = \text{Yaw}(\psi) &= \begin{bmatrix}\cos(\psi) & -\sin(\psi) & 0\\ \sin(\psi) & \cos(\psi) & 0\\ 0 & 0 & 1\end{bmatrix}
\end{aligned} \tag{43}$$

8 3D Rotation matrix from Euler angles

Euler angles can be applied sequentially in one of the two ways:

1. Proper Euler angles (z-x-z, x-y-x, y-z-y, z-y-z, x-z-x, y-x-y)
2. Tait-Bryan angles (x-y-z, y-z-x, z-x-y, x-z-y, z-y-x, y-x-z).

One of the most common application of Euler angles is X-Y-Z:

$$R(\phi,\theta,\psi) = R_z(\psi)R_y(\theta)R_x(\phi) = \text{Yaw}(\psi)\text{Pitch}(\theta)\text{Roll}(\phi). \tag{44}$$

Note that the rotation matrix application is read from right to left.

$$R(\phi,\theta,\psi) = \begin{bmatrix} c_\psi & -s_\psi & 0 \\ s_\psi & c_\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_\theta & 0 & s_\theta \\ 0 & 1 & 0 \\ -s_\theta & 0 & c_\theta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_\phi & -s_\phi \\ 0 & s_\phi & c_\phi \end{bmatrix}$$
$$= \begin{bmatrix} c_\psi & -s_\psi & 0 \\ s_\psi & c_\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_\theta & s_\theta s_\phi & s_\theta c_\phi \\ 0 & c_\phi & -s_\phi \\ -s_\theta & c_\theta s_\phi & c_\theta c_\phi \end{bmatrix}$$
$$= \begin{bmatrix} c_\psi c_\theta & c_\psi s_\theta s_\phi - s_\psi c_\phi & c_\psi s_\theta c_\phi + s_\psi s_\phi \\ s_\psi c_\theta & s_\psi s_\theta s_\phi + c_\psi c_\phi & s_\psi s_\theta c_\phi - c_\psi s_\phi \\ -s_\theta & c_\theta s_\phi & c_\theta c_\phi \end{bmatrix} \quad (45)$$

For a given 3D rotation matrix R, whose elements are r_{ij} as follows

$$R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}, \quad (46)$$

the roll, pitch, yaw angles can be read as,

$$\phi = \arctan2(r_{32}, r_{33}) \quad (47)$$
$$\theta = -\arcsin(r_{31}) \quad (48)$$
$$\psi = \arctan2(r_{22}, r_{21}) \quad (49)$$

8.1 Gimbal lock

When pitch $\theta = \frac{\pi}{2}$, then yaw-axis (Z-axis) coincides with roll-axis (X-axis). In such a case, inversion from a rotation matrix leads to infinitely possible solutions, because $c_\theta = 0$ and that leads to $r_{32} = r_{33} = r_{22} = r_{21} = 0$.

8.2 Orthogonality and determinant

Let 3D rotation be represented by a block matrix of 2D rotation $R_2(\phi)$.

$$R_x(\phi) = \begin{bmatrix} 1 & \mathbf{0}^\top \\ \mathbf{0} & R_2(\phi) \end{bmatrix}$$

$$R_x^\top(\phi) R_x(\phi) = \begin{bmatrix} 1 & \mathbf{0}^\top \\ \mathbf{0} & R_2(\phi) \end{bmatrix}^\top \begin{bmatrix} 1 & \mathbf{0}^\top \\ \mathbf{0} & R_2(\phi) \end{bmatrix}$$
$$= \begin{bmatrix} 1 & \mathbf{0}^\top \\ \mathbf{0} & R_2^\top(\phi) R_2(\phi) \end{bmatrix} = I \quad (50)$$

Same check can be applied to $R_y(\theta)$ and $R_z(\psi)$ as well. If two matrices A and B are orthogonal, then AB is also orthogonal:

$$(AB)^\top (AB) = B^\top A^\top AB = B^\top I B = B^\top B = I. \quad (51)$$

Hence any combination of principal rotations is also orthogonal.

Similar procedure can be followed to establish that $\det(R) = 1$.

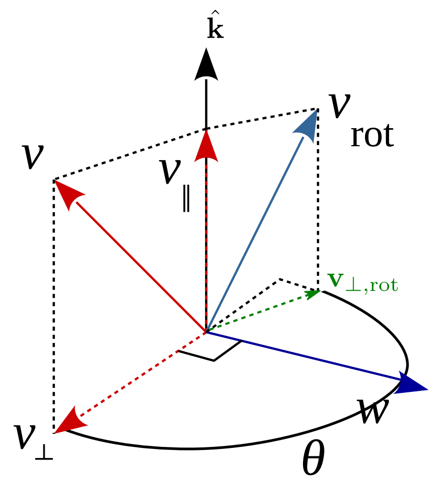
9 3D Transformation matrix

For 3D systems transformation matrix is 4×4 matrix denoted by ${}^wT_c$,

$$\underline{\mathbf{p}}_w = \begin{bmatrix} R(\phi,\theta,\psi) & {}^w\mathbf{t}_c \\ \mathbf{0}^\top & 1 \end{bmatrix} \underline{\mathbf{p}}_c = {}^wT_c \underline{\mathbf{p}}_c, \quad (52)$$

where ${}^w\mathbf{t}_c \in \mathbb{R}^3$, $\underline{\mathbf{p}}_c = \begin{bmatrix} \mathbf{p}_c \\ 1 \end{bmatrix}$ and $\mathbf{p}_c \in \mathbb{R}^3$.

10 Axis-angle representation



Cross product gives us a vector that is orthogonal to the plane of two vectors, let $\mathbf{w} = \hat{\mathbf{k}} \times \mathbf{v}$ be such a vector. Note that the magnitude of $\mathbf{w}$, $\|\mathbf{w}\| = \|\hat{\mathbf{k}}\|\mathbf{v}\sin(\phi)$, where ϕ is the angle between the unit-vector $\hat{\mathbf{k}}$ and $\mathbf{v}$.

$$\mathbf{v}_\perp = -\hat{\mathbf{k}} \times \mathbf{w} = -\hat{\mathbf{k}} \times (\hat{\mathbf{k}} \times \mathbf{v}) \tag{53}$$

$$\begin{aligned}
\mathbf{v}_{\perp,\text{rot}} &= \mathbf{v}_\perp \cos(\theta) + \mathbf{w}\sin(\theta) \qquad (54)\\
\mathbf{v}_{\text{rot}} &= \mathbf{v}_\parallel + \mathbf{v}_{\perp,\text{rot}}\\
&= \mathbf{v} - \mathbf{v}_\perp + \mathbf{v}_\perp \cos(\theta) + \mathbf{w}\sin(\theta)\\
&= \mathbf{v} - (1-\cos(\theta))\mathbf{v}_\perp + \mathbf{w}\sin(\theta)\\
&= \mathbf{v} + (1-\cos(\theta))\hat{\mathbf{k}} \times (\hat{\mathbf{k}} \times \mathbf{v}) + \sin(\theta)\hat{\mathbf{k}} \times \mathbf{v} \qquad (55)
\end{aligned}$$

Define cross product matrix K of $\hat{\mathbf{k}} = \begin{bmatrix} k_x \\ k_y \\ k_z \end{bmatrix}$ as,

$$K = \begin{bmatrix} 0 & -k_z & k_y \\ k_z & 0 & -k_x \\ -k_y & k_x & 0 \end{bmatrix}. \tag{56}$$

$$\begin{aligned}
\mathbf{v}_{\text{rot}} &= \mathbf{v} + (1-\cos(\theta))\hat{\mathbf{k}} \times (\hat{\mathbf{k}} \times \mathbf{v}) + \sin(\theta)\hat{\mathbf{k}} \times \mathbf{v}\\
&= (I + (1-\cos(\theta))K^2 + \sin(\theta)K)\mathbf{v} \qquad (57)
\end{aligned}$$

Thus the rotation matrix corresponding to axis-angle $\theta, \hat{\mathbf{k}}$ is given by,

$$R(\theta, \hat{\mathbf{k}}) = I + \sin(\theta)K + (1-\cos(\theta))K^2 \tag{58}$$

To get back θ and $\hat{\mathbf{k}}$ from R, first note that,

$$K^2 = \begin{bmatrix} -k_z^2 - k_y^2 & k_x k_y & k_z k_x \\ k_x k_y & -k_x^2 - k_z^2 & k_z k_y \\ k_x k_z & k_y k_z & -k_x^2 - k_y^2 \end{bmatrix} \tag{59}$$

Also we can use trace to separate θ from axis,

$$\begin{aligned}
\operatorname{tr}(R) &= \operatorname{tr}(I) + \sin(\theta)\operatorname{tr}(K) + (1-\cos(\theta))\operatorname{tr}(K^2)\\
&= 3 + 0 + (1-\cos(\theta))(-2(k_x^2 + k_y^2 + k_z^2)).\\
&= 3 - 2 + 2\cos(\theta) \qquad (60)
\end{aligned}$$

Thus we get $\theta = \arccos(\frac{\operatorname{tr}(R)-1}{2})$. We can estimate axis of rotation as the eigenvector corresponding eigenvalue 1, because $R\hat{\mathbf{k}} = \hat{\mathbf{k}}$.

Problem 1. *Degrees of Freedom of a quantity is the number independent scalar variables needed to represent that quantity. What is degrees of freedom required to*

1. *Position and orientation in 1-D*
2. *Position and orientation in 2-D*
3. *Position and orientation in 3-D*
4. *Position and orientation in 4-D*

(10 marks. Estimated time: 15 min) Justify your answer.

Problem 2. *Write a program in C++ that checks if a given 3x3 matrix is a valid Rotation matrix is a valid Rotation matrix (check for orthonormality i.e. orthogonality and determinant = 1). You may use Eigen's matrix multiplication and determinant() function. (10 marks. Used in the following problems. Estimated time: 15 min).*

Problem 3. *In class, we proved the expression to convert roll (θ), pitch (ϕ), yaw (ψ) from Euler Angles to Rotation matrix,*

$$R(\theta, \phi, \psi) = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} = R_z(\psi)R_y(\phi)R_x(\theta). \tag{61}$$

What if we want to do the inverse? Prove that given a proper 3x3 rotation matrix ($R^\top R = I$ and $det(R) = 1$) , the Euler angles are given by

$$\begin{bmatrix} \theta(R) \\ \phi(R) \\ \psi(R) \end{bmatrix} = \begin{bmatrix} \arctan2(r_{32}, r_{33}) \\ -\arcsin(r_{31}) \\ \arctan2(r_{21}, r_{11}) \end{bmatrix} \tag{62}$$

where r_{ij} is the element in ith row and jth column of the rotation matrix R. (10 marks. Used in the following problems. Estimated time: 15 min).

Problem 4. *Write a pair of functions in C++ that converts rotation matrix from XYZ Euler angles (roll, pitch, yaw) and vice versa. Test the pair of functions with randomly generated Euler angles. And check if the converted rotation matrix is orthonormal. What* *happens when pitch = $\pi/2$, are you able to convert from rotation matrix to Euler angle? Why or why not? (50 marks. Estimated time: 30 min)*

Problem 5. *Write a function in C++ that generates a 4x4 transformation matrix given XYZ Euler angles (roll, pitch, yaw) and translation. You can use the function that you wrote in Prob 7(20 marks. Estimated time: 15 min).*

Problem 6. *In class we proved the Rodrigues formula that converts from axis-angle representation $(\theta, \hat{\mathbf{k}})$, where θ is the angle of rotation and $\hat{\mathbf{k}}$ is the axis of rotation ($\|\hat{\mathbf{k}}\| = 1$). Let $\mathbf{K} = [\hat{\mathbf{k}}]_\times$ be the cross product matrix of $\hat{\mathbf{k}}$. The cross product matrix of $\hat{\mathbf{k}} = [k_x, k_y, k_z]^\top$ (such that $k_x^2 + k_y^2 + k_z^2 = 1$) is defined as,*

$$\mathbf{K} = [\hat{\mathbf{k}}]_\times = \begin{bmatrix} 0 & -k_z & k_y \\ k_z & 0 & -k_x \\ -k_y & k_x & 0 \end{bmatrix} \tag{63}$$

The corresponding rotation matrix is given by,

$$R(\theta, \hat{\mathbf{k}}) = \mathbf{I} + \sin\theta \mathbf{K} + (1 - cos\theta)\mathbf{K}^2. \tag{64}$$

An exponential of a square matrix $\mathbf{M}$ is defined as

$$\exp(\mathbf{M}) = \sum_{n=0}^{\infty} \frac{1}{n!}\mathbf{M}^k = \mathbf{I} + \frac{1}{1!}\mathbf{M} + \frac{1}{2!}\mathbf{M}^2 + \dots \tag{65}$$

Recall the series expansion of $\sin\theta$, and $\cos\theta$,

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \dots \tag{66}$$

$$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \dots \tag{67}$$

1. *First prove that $\mathbf{K}^3 = -\mathbf{K}$. (15 marks, 15 minutes)*
2. *As a result note that $\mathbf{K}^4 = -\mathbf{K}^2$, $\mathbf{K}^5 = \mathbf{K}$, and so on. In general, $\mathbf{K}^{2n+1} = (-1)^n\mathbf{K}$ and $\mathbf{K}^{2n+2} = (-1)^n\mathbf{K}^2$. Using the expansion of $\sin\theta$ and $\cos\theta$, prove that $R(\theta, \hat{\mathbf{k}}) = \exp(\theta\mathbf{K})$. (30 marks, 30 minutes)*

Problem 7. *Write a pair of functions in C++ that converts rotation matrix from axis-angle representation and vice versa. Recall that*

$$R(\theta, \hat{\mathbf{k}}) = \mathbf{I} + \sin\theta \mathbf{K} + (1 - cos\theta)\mathbf{K}^2. \tag{68}$$

and to get axis-angle back from a given rotation matrix

$$R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}, \tag{69}$$

we have

$$\theta = \cos^{-1}\left(\frac{tr(R) - 1}{2}\right) \tag{70}$$

$$\hat{\mathbf{k}} = \frac{1}{2\sin\theta}\begin{bmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{bmatrix} \quad \textit{if } \theta \neq 0 \textit{ or } \pi. \tag{71}$$

If $\theta = 0$ or π, then

$$\hat{\mathbf{k}} = \pm \begin{bmatrix} \sqrt{(r_{11} + 1)/2} \\ \sqrt{(r_{22} + 1)/2} \\ \sqrt{(r_{33} + 1)/2} \end{bmatrix} \tag{72}$$

(30 marks. Estimated time: 30 min)

Problem 8. *Recall the definition of Denavit-Hartenberg parameters from the video. Recall that transformation between two joints for the defined parameters d, θ, r, α is given by,*

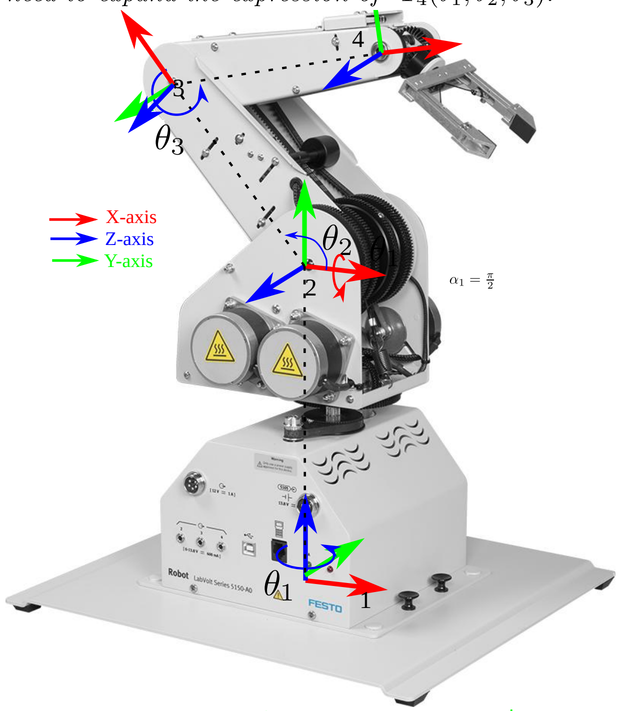
$$T = T_z(\theta, d)T_x(\alpha, r), \tag{73}$$

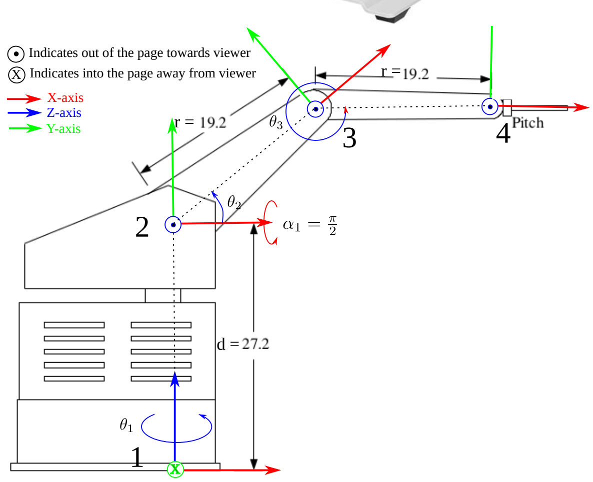
where

$$T_x(\alpha, r) = \begin{bmatrix} 1 & 0 & 0 & r \\ 0 & \cos(\alpha) & -\sin(\alpha) & 0 \\ 0 & \sin(\alpha) & \cos(\alpha) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{74}$$

$$T_z(\theta, d) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 & 0 \\ \sin(\theta) & \cos(\theta) & 0 & 0 \\ 0 & 0 & 1 & d \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{75}$$

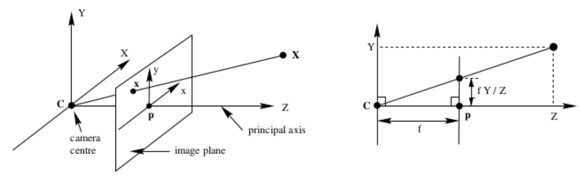
For the robot given below find transformation matrix from joint 4 to joint 1 assuming the joint angles *to be θ_1, θ_2, θ_3 respectively. Write the expression for ${}^3T_4(\theta_3)$, ${}^2T_3(\theta_2)$, ${}^1T_2(\theta_1)$ and then ${}^1T_4(\theta_1, \theta_2, \theta_3)$ in terms of the first three transformations. You do not need to expand the expression of ${}^1T_4(\theta_1, \theta_2, \theta_3)$.*





(15 marks. 15 min)

11 Camera projection model



$$\mathbf{K} = \begin{bmatrix} f_x & s & u_0 \\ 0 & f_y & v_0 \\ 0 & 0 & 1 \end{bmatrix} \tag{76}$$

$$\mathbf{u} = \begin{bmatrix} u \\ v \end{bmatrix} \text{ image coordinates in pixels} \tag{77}$$

$$\mathbf{X} = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} \text{ 3D coordinates in world units} \tag{78}$$

$$\underline{\mathbf{u}} = \begin{bmatrix} \mathbf{u} \\ 1 \end{bmatrix} \tag{79}$$

$$\lambda \underline{\mathbf{u}} = K\mathbf{X}, \text{ where } \lambda \neq 0 \tag{80}$$

12 Linear least squares or Pseudo-inverse

Pseudo-inverse of a matrix $\mathbf{A}$ is defined as a matrix $\mathbf{A}^\dagger$, such that $\mathbf{A}\mathbf{A}^\dagger\mathbf{A} = \mathbf{A}$.

$$\text{if } \mathbf{A} \text{ is tall and full-col rank, then } \mathbf{A}^\dagger = (\mathbf{A}^\top\mathbf{A})^{-1}\mathbf{A}^\top \tag{81}$$

$$\text{if } \mathbf{A} \text{ is fat and full-row rank, then } \mathbf{A}^\dagger = \mathbf{A}^\top(\mathbf{A}\mathbf{A}^\top)^{-1} \tag{82}$$

[1]

[1] See Appendix A of Gilbert Strang (1988): Linear Algebra and Its Applications

$$
\begin{aligned}
&\min_{\mathbf{x}} \|A\mathbf{x} - \mathbf{b}\|^2 && (83)\\
&= \min_{\mathbf{x}} (A\mathbf{x} - \mathbf{b})^\top (A\mathbf{x} - \mathbf{b}) && (84)\\
&= \min_{\mathbf{x}} (\mathbf{x}^\top A^\top - \mathbf{b}^\top)(A\mathbf{x} - \mathbf{b}) && (85)\\
&= \min_{\mathbf{x}} (\mathbf{x}^\top A^\top - \mathbf{b}^\top)(A\mathbf{x} - \mathbf{b}) && (86)\\
&= \min_{\mathbf{x}} \mathbf{x}^\top A^\top A\mathbf{x} - \mathbf{b}^\top A\mathbf{x} - \mathbf{x}^\top A^\top \mathbf{b} + \mathbf{b}^\top \mathbf{b} && (87)
\end{aligned}
$$

Recall that a minimum (or maximum) point of a differentiable function $f(\mathbf{x})$, $f'(\mathbf{x}) = 0$. Let us define vector derivative as

$$
\frac{\partial f(\mathbf{x})}{\partial \mathbf{x}} = \begin{bmatrix} \frac{\partial f(\mathbf{x})}{\partial x_1} \\ \frac{\partial f(\mathbf{x})}{\partial x_2} \\ \vdots \\ \frac{\partial f(\mathbf{x})}{\partial x_n} \end{bmatrix} \tag{88}
$$

You can verfiy that

$$
\frac{\partial}{\partial \mathbf{x}} \mathbf{x}^\top Q \mathbf{x} = 2Q\mathbf{x} \tag{89}
$$

$$
\frac{\partial}{\partial \mathbf{x}} \mathbf{b}^\top \mathbf{x} = \mathbf{b} \tag{90}
$$

At a minimum point $\mathbf{x}$,

$$
\frac{\partial}{\partial \mathbf{x}} \left(\mathbf{x}^\top A^\top A\mathbf{x} - \mathbf{b}^\top A\mathbf{x} - \mathbf{x}^\top A^\top \mathbf{b} + \mathbf{b}^\top \mathbf{b} \right) = 0 \tag{91}
$$

Note that $\mathbf{b}^\top A\mathbf{x}$ is a scalar, and hence $\mathbf{b}^\top A\mathbf{x} = (\mathbf{b}^\top A\mathbf{x})^\top = \mathbf{x}^\top A^\top \mathbf{b}$.

$$
\implies 2A^\top A\mathbf{x} - 2A^\top \mathbf{b} = 0 \tag{92}
$$

$$
\implies \mathbf{x} = \underbrace{(A^\top A)^{-1} A^\top}_{A^\dagger} \mathbf{b} \tag{93}
$$

List of Theorems