ECE 417/598: Null space, Singular Value Decompsition

Vikas Dhiman.

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Homogeneous representation of lines

$$\mathbb{P}^2 = \mathbb{R}^3 - \{(0,0,0)^\top\}$$

$$ax + by + 1.c = 0$$

$$\mathbf{l} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$\mathbf{x} = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

The point $\mathbf{x} \in \mathbb{P}^2$ lies on a line $\mathbf{l}$ if and only if

$$\mathbf{l}^\top \mathbf{x} = 0$$

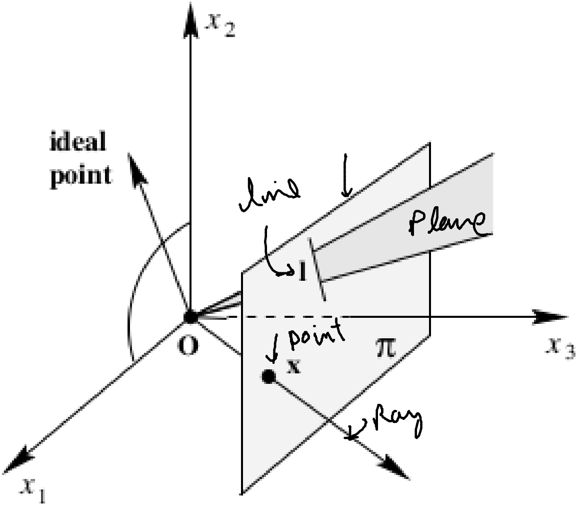
$x \in \mathbb{R}^3$ $\quad \underline{x} = \lambda \underline{x}$

$\lambda \neq 0$

2 DOF

$$\begin{bmatrix} 2a \\ 2b \\ 2c \end{bmatrix}$$

Points are rays and lines are planes



Intersection of lines

Two line $\mathbf{l}_1$ and $\mathbf{l}_2$ intersect at $\mathbf{x} \in \mathbb{P}^2$

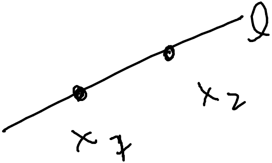
$$\mathbf{x} = \mathbf{l}_1 \times \mathbf{l}_2$$

$$l_1^T x = 0$$
$$l_2^T x = 0$$

Line joining points

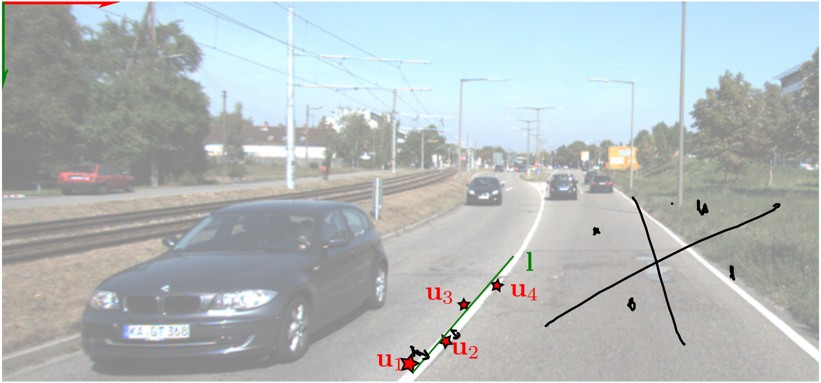
Two point $\mathbf{x}_1$ and $\mathbf{x}_2$ form a $\mathbf{l} \in \mathbb{P}^2$

$$\mathbf{l} = \mathbf{x}_1 \times \mathbf{x}_2$$



$$l^T x_1 = 0$$

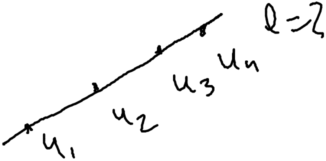
$$l^T x_2 = 0$$



$y = mx + c$

$ax + by + c = 0$

$$\underline{\mathbf{u}}_1 = [100, 98, 1]^\top$$
$$\underline{\mathbf{u}}_2 = [105, 95, 1]^\top$$
$$\underline{\mathbf{u}}_3 = [107, 90, 1]^\top$$
$$\underline{\mathbf{u}}_4 = [110, 85, 1]^\top$$



$l = u_1 \times u_2$

Find the line $\mathbf{l}$ such that it is the "closest line" passing through $\mathbf{u}_1, \ldots, \mathbf{u}_4$.

$$U = \begin{bmatrix} \mathbf{u}_1^\top \\ \mathbf{u}_2^\top \\ \mathbf{u}_3^\top \\ \mathbf{u}_4^\top \end{bmatrix}$$

$u_1^T l = 0$

$u_2^T l = 0$

$u_3^T l = 0$

$u_4^T l = 0$

We want to solve for $\mathbf{l}$ such that

$$U\mathbf{l} = 0$$

$$Ax = 0$$

$x = 0$

Vector space?

$\{v_1, v_2, v_3 \ldots v_n\}$

vector

$\to u = \lambda_1 v_1 + \lambda_2 v_2 + \lambda_3 v_3 + \ldots + \lambda v_n$

$\lambda_1, \lambda_2, \ldots \lambda_n \in \mathbb{R}$

vectors

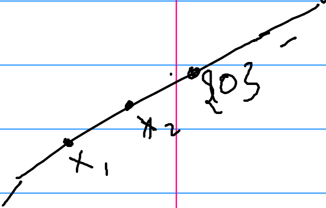
$\{0\}$

Examples of vector space

① $\{\underline{0}\}$ $\quad \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}_n$

is a vector space

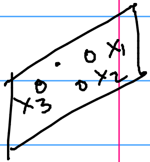
② A line (through origin) is a vector space



$$\underline{x} = \lambda_1 \underline{x}_1 + \lambda_2 \underline{x}_2$$

$$\underline{x} = \lambda \underline{x}_1 + (1-\lambda) \underline{x}_2$$

(3) A plane through origin is a vector space



$$\underline{x} = a\underline{x}_1 + b\underline{x}_2 + c\underline{x}_3$$

$\mathbb{R}^n$ ~~$\mathbb{R}^2$ $\mathbb{R}^3$~~

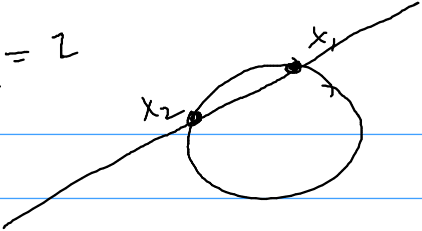
(4)

$A, B \in \mathbb{R}^{2\times 3}$ $\quad C = \lambda_1 A + \lambda_2 B$

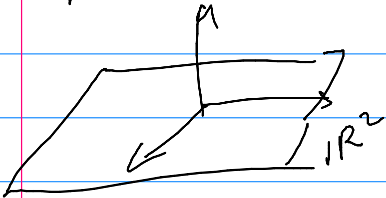
$C \in \mathbb{R}^{2\times 3}$

(b) Not a vector space

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$



sub spaces



$\mathbb{R}^3$

Example

a plane in 3D

#

Column shace of a matrix $A \in \mathbb{R}^{m \times n}$

$$A = m \begin{bmatrix} \underline{a}_1 & \underline{a}_2 & \underline{a}_3 & \cdots & \underline{a}_n \end{bmatrix}$$

$\longleftarrow n \longrightarrow$

column shace = $\lambda_1 \underline{a}_1 + \dots + \lambda_n a_n$

spanned

$A \in \mathbb{R}^{m \times n}$

$x \in \mathbb{R}^n$

$$A \underline{x} = \underline{b}$$

$b \in \mathbb{R}^m$

Column space $\subseteq \mathbb{R}^m$ = Set of all possible $\underline{b}$ with an exact solution

$\underline{\text{Null space}}$ $\subseteq \mathbb{R}^n$ = Set of all possible $\underline{x}$ such that

$$A\underline{x} = 0$$

The column space (also called the range) of matrix $A \in \mathbb{R}^{m \times n}$, denoted by $\mathcal{R}(A)$ is defined as the set of all vectors $\mathbf{b} \in \mathbb{R}^m$ that can be generated by $\mathbf{b} = A\mathbf{x}$ where $\mathbf{x} \in \mathbb{R}^n$, that is,

$$\mathcal{R}(A) = \{\mathbf{b} \mid \mathbf{b} = A\mathbf{x} \text{ for all } \mathbf{x} \in \mathbb{R}^n\}. \tag{1}$$

The nullspace of $A \in \mathbb{R}^{m \times n}$ is defined as the set of all vectors $\mathbf{x} \in \mathbb{R}^n$ such that $A\mathbf{x} = \mathbf{0}_m$. In other words,

$$\mathcal{N}(A) = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{0}_m = A\mathbf{x}\}. \tag{2}$$

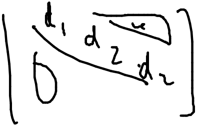
The task of finding the column space or the null space is the task of finding the minimal set of vectors that *span* the vector spaces $\mathcal{R}(A)$ or $\mathcal{N}(A)$ respectively.

Find the $\mathcal{R}(A)$ and $\mathcal{N}(A)$ of the matrix A

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 2 & 4 & 6 \end{bmatrix} = \begin{bmatrix} \mathbf{r}_1^\top \\ \mathbf{r}_2^\top \\ \mathbf{r}_3^\top \end{bmatrix} \Rightarrow Ax = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$0 \rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 2 & 4 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

upper triangula



$$r_2^{T'} = r_2^T - 4 r_1^T$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 2 & 4 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 - 4b_1 \\ b_3 \end{bmatrix}$$

$$r_3^{T\prime} = r_3^T - 2 r_1^T$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 - 4b_1 \\ b_3 - 2b_1 \end{bmatrix}$$

Subspaces = represent in terms of basis vectors (linear independent)

$$= \lambda_1 \underline{v}_1 + \lambda_2 \underline{v}_2 + \lambda_3 v_3$$

$$Ax = 0$$

Null space

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$x_1 + 2x_2 + 3x_3 = 0$$

$$-3x_2 - 6x_3 = 0$$

$$x_2 = -2x_3$$

$$x_1 - 4x_3 + 3x_3 = 0$$

$$x_1 = x_3$$

$$\underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_3 \\ -2x_3 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} x_3$$

$$N(A) = \lambda_1 \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

<u>Columnspace</u> $A \rightarrow$ Upper triangular

$A = LU$ decomposition

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 2 & 4 & 6 \end{bmatrix} =$$

row2' = row2 − 4 row 1

$$\begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 2 & 4 & 6 \end{bmatrix}$$

row2 = row2' + 4 row1

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 2 & 4 & 6 \end{bmatrix}$$

lower Δ → LU ← upper Δ

LU factorization

$row3' = row3 - 2\,row1$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & 0 & 0 \end{bmatrix}$$

LDU

$$= \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

rank (A) = Dimensionality of its column space
= " " " row space

$$R(A) = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & -3 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 4 & 5 \\ 2 & 8 \end{bmatrix}$$

$= \lambda_1 \underline{v_1} + \lambda_2 v_2$ (columns: v_1, v_2)

Eigenvalues and Eigenvectors

For a square matrix A, the λ_i and $\mathbf{x}_i$ that satisfy the following equation are called eigenvalues and eigenvectors respectively.

$$A\mathbf{x} = \lambda\mathbf{x} \text{ or } (A - \lambda I)\mathbf{x} = 0 \tag{3}$$

λ is chosen to ensure that $A - \lambda I$ has null space, hence, characteristic equation

$$\det(A - \lambda I) = 0 \tag{4}$$

For symmetrix matrix $A = A^\top$, eigenvalues are real, and eigenvectors are orthonormal,

$$A[\mathbf{x}_1, \ldots, \mathbf{x}_n] = [\mathbf{x}_1, \ldots, \mathbf{x}_n] \begin{bmatrix} \lambda_1 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & \lambda_n \end{bmatrix} \tag{5}$$

$$AS = S\Lambda \tag{6}$$

$$\text{if } A = A^\top \text{ then } A = S\Lambda S^\top \tag{7}$$

We introduce two vocabulary words to describe what we have seen. Let A be a square matrix and a scalar.

- The geometric multiplicity of is the dimension of the -eigenspace. In other words, $\dim \mathcal{N}((AI))$. ▪ The algebraic multiplicity of is the number of times (t) occurs as a factor of $\det(AtI)$.

Numerical example

Singular Value Decomposition (SVD)

$$A = U \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} V^\top \tag{8}$$

$$A^\top A = V \Sigma^2 V^{-1} \tag{9}$$

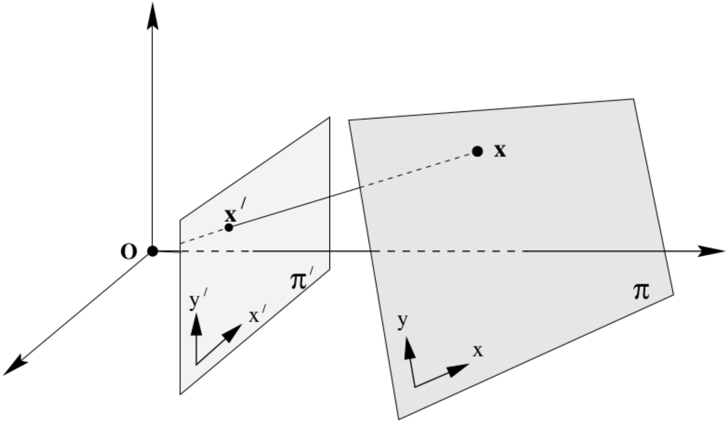
$$A^\top A \mathbf{v}_i = \lambda_i \mathbf{v}_i \qquad \lambda_i = \sigma_i^2 \tag{10}$$

$$AV = U \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} \tag{11}$$

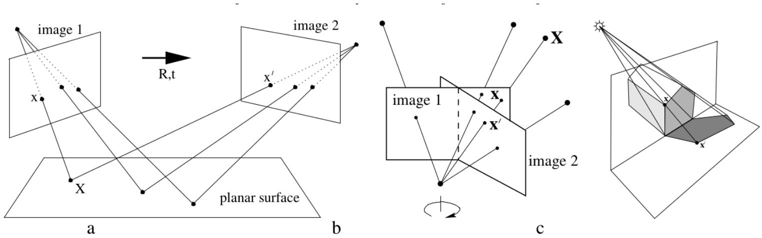
$$U^+ = \Sigma^{-1} A V^+ \tag{12}$$

Numerical example

Homography



Examples of Homography




Top view
Graphic
Look! Safe to move?
Settings

Computing Homography

Computing Homography

Solving for Homography derivation

Direct Linear Transformation (DLT) algorithm

Objective

Given $n \geq 4$ 2D to 2D point correspondences $\{\mathbf{x}_i \leftrightarrow \mathbf{x}'_i\}$, determine the 2D homography matrix $\mathtt{H}$ such that $\mathbf{x}'_i = \mathtt{H}\mathbf{x}_i$.

Algorithm

(i) For each correspondence $\mathbf{x}_i \leftrightarrow \mathbf{x}'_i$ compute the matrix $\mathtt{A}_i$ from (4.1). Only the first two rows need be used in general.

(ii) Assemble the n 2×9 matrices $\mathtt{A}_i$ into a single $2n \times 9$ matrix $\mathtt{A}$.

(iii) Obtain the SVD of $\mathtt{A}$ (section A4.4(p585)). The unit singular vector corresponding to the smallest singular value is the solution $\mathbf{h}$. Specifically, if $\mathtt{A} = \mathtt{UDV}^\mathsf{T}$ with $\mathtt{D}$ diagonal with positive diagonal entries, arranged in descending order down the diagonal, then $\mathbf{h}$ is the last column of $\mathtt{V}$.

(iv) The matrix $\mathtt{H}$ is determined from $\mathbf{h}$ as in (4.2).

2D homography

Given a set of points $\mathbf{x}_i \in \mathbb{P}^2$ and a corresponding set of points $\mathbf{x}'_i \in \mathbb{P}^2$, compute the projective transformation that takes each $\mathbf{x}_i$ to $\mathbf{x}'_i$. In a practical situation, the points $\mathbf{x}_i$ and $\mathbf{x}'_i$ are points in two images (or the same image), each image being considered as a projective plane $\mathbb{P}^2$.

3D to 2D camera projection matrix estimation

Given a set of points $\mathbf{X}_i$ in 3D space, and a set of corresponding points $\mathbf{x}_i$ in an image, find the 3D to 2D projective $\mathbf{P}$ mapping that maps $\mathbf{X}_i$ to $\mathbf{x}_i = \mathbf{P}\mathbf{X}_i$.