# ECE 417/598: Null space, Singular Value Decompsition

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# Homogeneous representation of lines

$$\mathbb{P}^2 = \mathbb{R}^3 - \{(0,0,0)^\top\}$$

$$ax + by + 1.c = 0$$

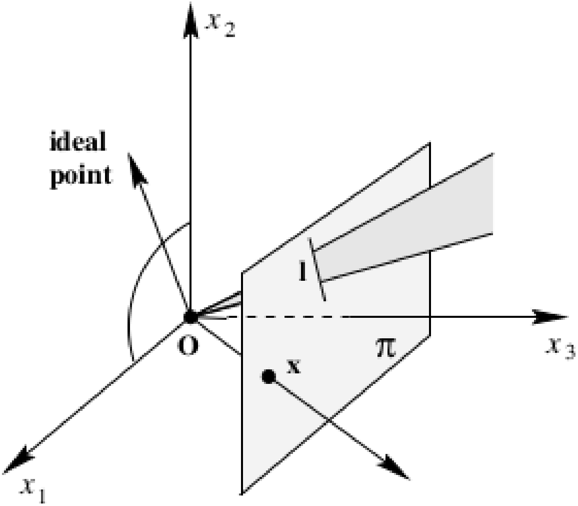
$$\mathbf{l} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$\mathbf{x} = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

The point $\mathbf{x} \in \mathbb{P}^2$ lies on a line $\mathbf{l}$ if and only if

$$\mathbf{l}^\top \mathbf{x} = 0$$

# Points are rays and lines are planes



# Intersection of lines

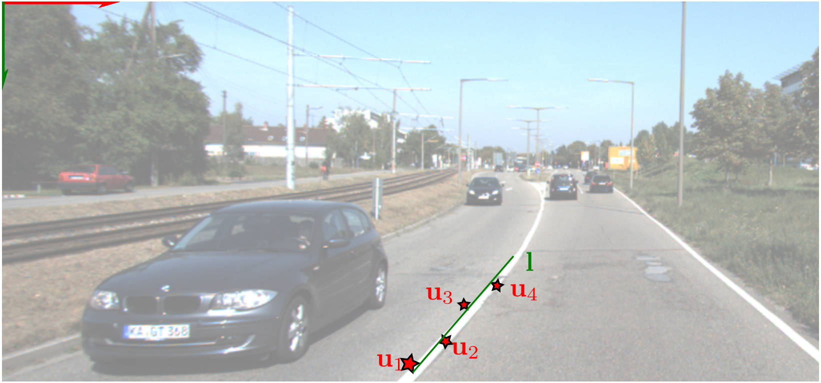
Two line $\mathbf{l}_1$ and $\mathbf{l}_2$ intersect at $\mathbf{x} \in \mathbb{P}^2$

$$\mathbf{x} = \mathbf{l}_1 \times \mathbf{l}_2$$

# Line joining points

Two point $\mathbf{x}_1$ and $\mathbf{x}_2$ form a $\mathbf{l} \in \mathbb{P}^2$

$$\mathbf{l} = \mathbf{x}_1 \times \mathbf{x}_2$$




$$\underline{\mathbf{u}}_1 = [100, 98, 1]^\top$$
$$\underline{\mathbf{u}}_2 = [105, 95, 1]^\top$$
$$\underline{\mathbf{u}}_3 = [107, 90, 1]^\top$$
$$\underline{\mathbf{u}}_4 = [110, 85, 1]^\top$$

Find the line $\mathbf{l}$ such that it is the "closest line" passing through $\mathbf{u}_1, \ldots, \mathbf{u}_4$.

$$U = \begin{bmatrix} \mathbf{u}_1^\top \\ \mathbf{u}_2^\top \\ \mathbf{u}_3^\top \\ \mathbf{u}_4^\top \end{bmatrix}$$

We want to solve for $\mathbf{l}$ such that

$$U\mathbf{l} = 0$$

The column space (also called the range) of matrix $A \in \mathbb{R}^{m \times n}$, denoted by $\mathcal{R}(A)$ is defined as the set of all vectors $\mathbf{b} \in \mathbb{R}^m$ that can be generated by $\mathbf{b} = A\mathbf{x}$ where $\mathbf{x} \in \mathbb{R}^n$, that is,

$$\mathcal{R}(A) = \{\mathbf{b} \mid \mathbf{b} = A\mathbf{x} \text{ for all } \mathbf{x} \in \mathbb{R}^n\}. \tag{1}$$

The nullspace of $A \in \mathbb{R}^{m \times n}$ is defined as the set of all vectors $\mathbf{x} \in \mathbb{R}^n$ such that $A\mathbf{x} = \mathbf{0}_m$. In other words,

$$\mathcal{N}(A) = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{0}_m = A\mathbf{x}\}. \tag{2}$$

The task of finding the column space or the null space is the task of finding the minimal set of vectors that *span* the vector spaces $\mathcal{R}(A)$ or $\mathcal{N}(A)$ respectively.

Find the $\mathcal{R}(A)$ and $\mathcal{N}(A)$ of the matrix $A$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 2 & 4 & 6 \end{bmatrix} = \begin{bmatrix} \mathbf{r}_1^\top \\ \mathbf{r}_2^\top \\ \mathbf{r}_3^\top \end{bmatrix}$$

$A = LDU$

$Ux = 0 \Rightarrow x \in N(U)$
$= N(A)$

$R(LDU[:, :2]) = R(A)$

rank of the matrix

$A^T = \underbrace{U^T}_{} D^T L^T$ ← upper triangular

low triangular

$$A = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}}_{U}$$

$$A^T = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 4 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R(A^T)$ = Row space of $A$

$N(A^T)$ = left null space of $A$ $\qquad Ax = 0$

$$A^T y = 0 \iff y^T A = 0^T$$

Four fundamental subspaces of matrix $A \in \mathbb{R}^{m \times n}$:

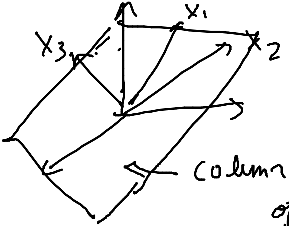
1. Column space: All possible values of $\mathbf{b} = A\mathbf{x}$ for any $\mathbf{x} \in \mathbb{R}^n$.
2. Null space: All possible values of $\mathbf{x} \in \mathbb{R}^n$ so that $A\mathbf{x} = \mathbf{0}_m$.
3. Row space: Column space of $A^\top$. All possible values of $\mathbf{b} = A^\top \mathbf{x}$ for any $\mathbf{x} \in \mathbb{R}^m$.
4. Left Null space: Null space of $A^\top$. All possible values of $\mathbf{y} \in \mathbb{R}^m$ so that $\mathbf{y}^\top \mathbf{A} = 0$.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 2 & 4 & 6 \end{bmatrix} = \begin{bmatrix} \underline{x}_1, & \underline{x}_2, & \underline{x}_3 \end{bmatrix}$$

$r_3 = 2r_1$

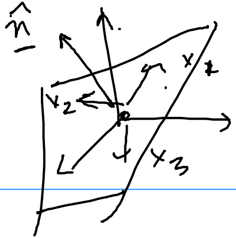
$r_3 = \lambda_1 r_1 + \lambda_2 r_2$

$R(A) = \lambda_1 \underline{x}_1 + \lambda_2 \underline{x}_2 + \lambda_3 \underline{x}_3$




columns space of matrix A

$$A = \begin{bmatrix} \underline{x}_1 & \underline{x}_2 & \underline{x}_3 \end{bmatrix}$$




$$\hat{n}^T \begin{bmatrix} \underline{x}_1 & \underline{x}_2 & \underline{x}_3 \end{bmatrix} = 0$$

Left null space

$\underline{x}_1 \times \underline{x}_2$

$$\underline{y}^T A = 0 \iff A^T \underline{y} = 0$$

$$y^T A = 0$$

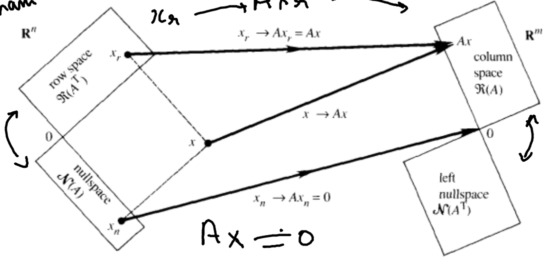
$$N(A^T) \perp R(A)$$

$$y^T (Ax) = 0$$

domain

$x_r \longrightarrow Ax_r \longrightarrow$

Range




$Ax = 0$

$f(x) \to y$ (domain, rang)

3

$A = [x_1 \;\; \lambda_1 x_1 \;\; \lambda_1 x_1]$

$= \lambda_1 x_1$

## The four fundamental subspaces of $A$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 2 & 4 & 6 \end{bmatrix}$$

## Geometric intuition

# Eigenvalues and Eigenvectors

For a square matrix $A$, the $\lambda_i$ and $\mathbf{x}_i$ that satisfy the following equation are called eigenvalues and eigenvectors respectively.

$$A\mathbf{x} = \lambda\mathbf{x} \text{ or } (A - \lambda I)\mathbf{x} = 0 \tag{3}$$

$\lambda$ is chosen to ensure that $A - \lambda I$ has null space, hence, characteristic equation

$$\det(A - \lambda I) = 0 \tag{4}$$

For symmetrix matrix $A = A^\top$, eigenvalues are real, and eigenvectors are orthonormal,

$$A[\mathbf{x}_1, \ldots, \mathbf{x}_n] = [\mathbf{x}_1, \ldots, \mathbf{x}_n] \begin{bmatrix} \lambda_1 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & \lambda_n \end{bmatrix} \tag{5}$$

$$AS = S\Lambda \tag{6}$$

$$\text{if } A = A^\top \text{ then } A = S\Lambda S^\top \tag{7}$$

$A\underline{v} = \lambda \underline{v}$

$\lambda$ = eigen value

$\underline{v}$ = eigen vector

$A \in \mathbb{R}^{n \times n}$

n - eigen values , n - eigen vector

maybe repeated repeated

may be complex

$A \underline{v}_1 = \lambda_1 v_1$

$A v_2 = \lambda_2 v_2$

$\vdots$

$A v_n = \lambda_n v_n$

$$\underbrace{A\begin{bmatrix} \underline{v}_1 & v_2 & \dots & v_n \end{bmatrix}} = \begin{bmatrix} \lambda_1 v_1, & \lambda_2 v_2 & \dots, & \lambda_n v_n \end{bmatrix}$$

$$AS = \begin{bmatrix} \underline{v}_1 & \dots & \underline{v}_n \end{bmatrix} \underbrace{\begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}}_{\Lambda}$$

$$= S\Lambda$$

$$\underline{A = S\Lambda S^{-1}}$$ eigen value decomposition

$$A^2 = AA = (S\Lambda S^{-1})(S\Lambda S^{-1})$$

$$= S\Lambda^2 S^{-1}$$

$$A^n = S\Lambda^n S^{-1}$$

eigen values / vector for small matrix

$$A\underline{v} = \lambda \underline{v}$$

$$Av = \lambda I v$$

$$\underline{(A-\lambda I)}\,\underline{v} = 0$$

A has full rank

then $\mathcal{N}(A) = \{0\}$

A to be NOT singular

$$\det(A - \lambda I) = 0$$

$$\det\left(\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 8 \\ 4 & 5 & 6 \end{bmatrix} - \begin{bmatrix} \lambda & & \\ & \lambda & 0 \\ 0 & & \lambda \end{bmatrix}\right) = 0$$

3-degree polynomial

$$\lambda_1, \lambda_2, \lambda_3$$

# Numerical example

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 2 & 4 & 6 \end{bmatrix}$$

Find eigen values and eigen vectors.

# Singular Value Decomposition (SVD)

$$A = U \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} V^\top \tag{8}$$

$$A^\top A = V \Sigma^2 V^{-1} \tag{9}$$

$$A^\top A \mathbf{v}_i = \lambda_i \mathbf{v}_i \qquad \lambda_i = \sigma_i^2 \tag{10}$$

$$AV = U \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} \tag{11}$$

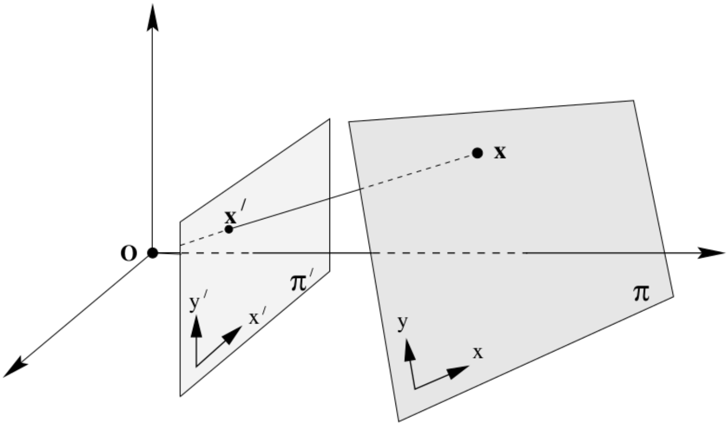
$$U^+ = \Sigma^{-1} A V^+ \tag{12}$$

# Numerical example

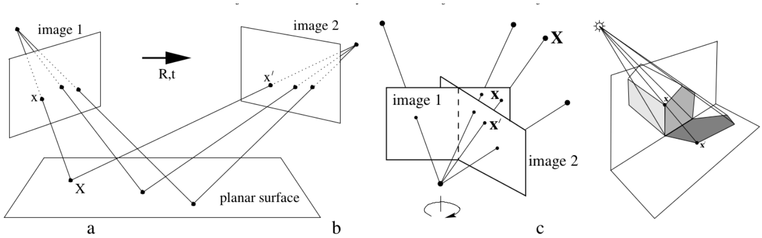
Find singular value decomposition

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 2 & 4 & 6 \end{bmatrix}$$

# Homography



# Examples of Homography




Top view
Graphic
Look! Safe to move?
Settings

# Computing Homography

# Computing Homography

# Solving for Homography derivation

# Direct Linear Transformation (DLT) algorithm

Objective

Given $n \geq 4$ 2D to 2D point correspondences $\{\mathbf{x}_i \leftrightarrow \mathbf{x}'_i\}$, determine the 2D homography matrix $\mathtt{H}$ such that $\mathbf{x}'_i = \mathtt{H}\mathbf{x}_i$.

Algorithm

(i) For each correspondence $\mathbf{x}_i \leftrightarrow \mathbf{x}'_i$ compute the matrix $\mathtt{A}_i$ from (4.1). Only the first two rows need be used in general.

(ii) Assemble the $n$ $2 \times 9$ matrices $\mathtt{A}_i$ into a single $2n \times 9$ matrix $\mathtt{A}$.

(iii) Obtain the SVD of $\mathtt{A}$ (section A4.4($p$585)). The unit singular vector corresponding to the smallest singular value is the solution $\mathbf{h}$. Specifically, if $\mathtt{A} = \mathtt{UDV}^\mathsf{T}$ with $\mathtt{D}$ diagonal with positive diagonal entries, arranged in descending order down the diagonal, then $\mathbf{h}$ is the last column of $\mathtt{V}$.

(iv) The matrix $\mathtt{H}$ is determined from $\mathbf{h}$ as in (4.2).

# 2D homography

Given a set of points $\mathbf{x}_i \in \mathbb{P}^2$ and a corresponding set of points $\mathbf{x}'_i \in \mathbb{P}^2$, compute the projective transformation that takes each $\mathbf{x}_i$ to $\mathbf{x}'_i$ . In a practical situation, the points $\mathbf{x}_i$ and $\mathbf{x}'_i$ are points in two images (or the same image), each image being considered as a projective plane $\mathbb{P}^2$.

# 3D to 2D camera projection matrix estimation

Given a set of points $\mathbf{X}_i$ in 3D space, and a set of corresponding points $\mathbf{x}_i$ in an image, find the 3D to 2D projective $\mathbf{P}$ mapping that maps $\mathbf{X}_i$ to $\mathbf{x}_i = \mathbf{P}\mathbf{X}_i$.