ECE 417/598: Eigen Value Decomposition, Singular Value Decompsition

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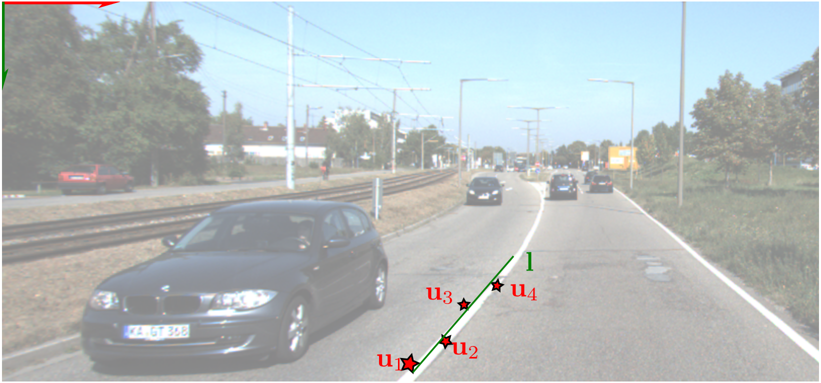


$$\underline{\mathbf{u}}_1 = [100, 98, 1]^\top$$
$$\underline{\mathbf{u}}_2 = [105, 95, 1]^\top$$
$$\underline{\mathbf{u}}_3 = [107, 90, 1]^\top$$
$$\underline{\mathbf{u}}_4 = [110, 85, 1]^\top$$

Find the line $\mathbf{l}$ such that it is the "closest line" passing through $\mathbf{u}_1, \ldots, \mathbf{u}_4$.

$$U = \begin{bmatrix} \mathbf{u}_1^\top \\ \mathbf{u}_2^\top \\ \mathbf{u}_3^\top \\ \mathbf{u}_4^\top \end{bmatrix}$$

We want to solve for $\mathbf{l}$ such that

$$U\mathbf{l} = 0$$

Eigenvalues and Eigenvectors

$x \to$ axis angle

For a square matrix A, the λ_i and $\mathbf{x}_i$ that satisfy the following equation are called eigenvalues and eigenvectors respectively.

x λx

$$A\mathbf{x} = \lambda\mathbf{x} \text{ or } (A - \lambda I)\mathbf{x} = 0 \tag{1}$$

$R\underline{x} = \lambda \underline{x}$

$R\underline{x} = \underline{x}$

λ is chosen to ensure that $A - \lambda I$ has null space, hence, characteristic equation

$$\det(A - \lambda I) = 0 \tag{2}$$

For symmetrix matrix $A = A^\top$, eigenvalues are real, and eigenvectors are orthonormal,

$$A[\mathbf{x}_1, \dots, \mathbf{x}_n] = [\mathbf{x}_1, \dots, \mathbf{x}_n] \begin{bmatrix} \lambda_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \lambda_n \end{bmatrix} \tag{3}$$

$$AS = S\Lambda \tag{4}$$

$$\text{if } A = A^\top \text{ then } A = S\Lambda S^\top \tag{5}$$

$$A\underline{x} = \lambda \underline{x}$$

linearly independent eigen vector

$A \in \mathbb{R}^{n \times n}$ $\quad v_1, \dots, v_n$

$$v_j \neq \underbrace{\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_{j-1} v_{j-1} + \alpha_{j+1} v_{j+1} + \dots + \alpha_n v_n}_{}$$

Linear combination

$$\Lambda = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$$

$\lambda_i \in \mathbb{R}$

$$A\underbrace{[v_1 \dots v_n]}_{S} = \underbrace{[v_1 \dots v_n]}_{S}\Lambda \Rightarrow A = S \Lambda S^{-1}$$

Numerical example

$$\det(A - \lambda I) = 0$$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 2 & 4 & 6 \end{bmatrix}$$

Find eigen values and eigen vectors Eigen library.
`https://github.com/wecacuee/ECE417-Mobile-Robots/blob/master/docs/slides/03-04-linear-algebra_files/findeig.cpp`

$A\underline{x} = \lambda \underline{x}$

if $\lambda = 0$, (eigen value $= 0$)

$A\underline{x} = 0$ $\quad \underline{x} \in N(A)$

eigen vector $\quad$ nullspace

Not all matrices possess n linearly independent eigenvectors, and therefore not all matrices are diagonalizable. The standard example of a "defective matrix" is

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$\underbrace{A = S \Lambda S^{-1}}$

diagonalizabili

$\det(A - \lambda I) = 0$

$\det \begin{bmatrix} -\lambda & 1 \\ 0 & -\lambda \end{bmatrix} = 0$

$\Rightarrow \lambda^2 - 0 = 0$

$\Rightarrow \lambda = 0, 0$

$Av = \lambda v$

if eigen vectors are linearly independent

$S = \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix}$

S^{-1} exist if its columns (or rows) are linearly independent

$\lambda_1 = 0$

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\begin{bmatrix} x_2 \\ 0 \end{bmatrix} = 0$$

$x_2 = 0$

$x_1 \in \mathbb{R}$

$\lambda_1 = 0$

$\underline{v}_1 = \begin{bmatrix} x_1 \\ 0 \end{bmatrix}$ for all $x_1 \in \mathbb{R}$

$\lambda_2 = 0,$

$\underline{v}_2 = \begin{bmatrix} y_1 \\ 0 \end{bmatrix}$ for al $y_1 \in \mathbb{R}$

If the eigenvectors $\underline{x_1}, \ldots, \underline{x_k}$ correspond to different eigenvalues $\lambda_1, \ldots, \lambda_k$ then those eigenvectors are linearly independent.

$$A\underline{x}_1 = \lambda_1 \underline{x}_1$$

$$A\underline{x}_2 = \lambda_2 \underline{x}_2 \qquad c_1, c_2 \in \mathbb{R}$$

$$\underline{x}_1 = c_2 \underline{x}_2 \iff c_1 \underline{x}_1 + c_2 \underline{x}_2 \neq 0$$

$$\lambda_1 \neq \lambda_2 \xrightarrow{\text{the}}$$

$$\frac{c_1}{\lambda_1} A \underline{x}_1 + \frac{c_2}{\lambda_2} A x_2 = c_1 \underline{x}_1 + c_2 x_2$$

Next time

$$A\left(\frac{c_1}{\lambda_1} x_1 + \frac{c_2}{\lambda_2} x_2\right) = 0$$

Find the eigen values and vectors of rotation matrix

$$R(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

$$\det(R - \lambda I) = 0$$

$$\det \begin{bmatrix} \cos(\theta) - \lambda & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) - \lambda \end{bmatrix} = 0$$

$$(\cos(\theta) - \lambda)^2 + \sin^2(\theta) = 0$$

$$(\cos(\theta) - \lambda)^2 = -\sin^2(\theta)$$

$$\cos(\theta) - \lambda = \pm i \sin(\theta)$$

$$\lambda = \cos(\theta) \pm i \sin(\theta) = e^{\pm i\theta}$$

Find the eigen values and vectors of rotation matrix

$$R(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{bmatrix}$$

$$\det[R - \lambda I] = 0$$

$$\det\begin{bmatrix} 1-\lambda & 0 & 0 \\ 0 & \cos\theta - \lambda & -\sin\theta \\ 0 & \sin\theta & \cos\theta - \lambda \end{bmatrix} = 0$$

$$\Rightarrow (1-\lambda)\det\begin{bmatrix} \cos\theta - \lambda & -\sin\theta \\ \sin\theta & \cos\theta - \lambda \end{bmatrix} = 0$$

$$\Rightarrow (1-\lambda)\left((\cos\theta - \lambda)^2 + \sin^2\theta\right) = 0$$

$$\Rightarrow \lambda = 1, \; e^{+i\theta}, \; e^{-i\theta}$$

$$Rv_1 = Iv_1 \Rightarrow (R-I)v_1 = 0$$

$$R\underline{v}_1 = \lambda_1 \underline{v}_1 \qquad \lambda_1 = 1$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 0 & 0 & 0 \\ 0 & \cos\theta - 1 & -\sin\theta \\ 0 & \sin\theta & \cos\theta - 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$x_1 = 1, \quad x_2 = 0, \quad x_3 = 0$ $\qquad x_1 \in \mathbb{R}$

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Compute the exponential of matrix $A = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}$ using the series expansion of $\exp(A)$ and the fact that $A^n = S\Lambda^n S^{-1}$.

$$\exp(A) = 1 + \frac{A}{1!} + \frac{A^2}{2!} + \frac{A^3}{3!} + \dots \infty$$

$$= 1 + \frac{S\Lambda S^{-1}}{1!} + \frac{S\Lambda^2 S^{-1}}{2!} + \dots \infty$$

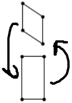
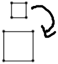
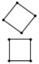
$$= S\left[1 + \frac{\Lambda}{1!} + \frac{\Lambda^2}{2!} + \dots \infty\right] S^{-1}$$

$$= S \exp(\Lambda) S^{-1}$$

$$\exp(A) = S \begin{bmatrix} e^{\lambda_1} & & \\ & \ddots & \\ & & e^{\lambda_n} \end{bmatrix} S^{-1}$$

Hierarchy of transforms

R,t

Group	Matrix	Distortion	Invariant properties
Projective 8 dof	$\begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix}$		Concurrency, collinearity, **order of contact**: intersection (1 pt contact); tangency (2 pt contact); inflections (3 pt contact with line); tangent discontinuities and cusps. cross ratio (ratio of ratio of lengths).
Affine 6 dof	$\begin{bmatrix} \boxed{\begin{matrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{matrix}} & \begin{matrix} t_x \\ t_y \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & 1 \end{bmatrix}$		Parallelism, ratio of areas, ratio of lengths on collinear or parallel lines (e.g. midpoints), linear combinations of vectors (e.g. centroids). The line at infinity, $\mathbf{l}_\infty$.
Similarity 4 dof	$\begin{bmatrix} sr_{11} & sr_{12} & t_x \\ sr_{21} & sr_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$		Ratio of lengths, angle. The circular points, **I**, **J** (see section 2.7.3).
Euclidean 3 dof	$\begin{bmatrix} r_{11} & r_{12} & t_x \\ r_{21} & r_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$		Length, area

$$\begin{bmatrix} y_1 \\ y_2 \\ 1 \end{bmatrix} = \begin{bmatrix} R_{2\times 2} & t_{2\times 1} \\ 0^T & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ 1 \end{bmatrix}$$

$$\underline{y} = R\underline{x} + t$$

$$\|\underline{y}\| = \|\underline{x}\|$$

(2) Similarity transfor

$s \in \mathbb{R}$

$$\underline{y} = sR\underline{x} + t$$

Singular Value Decomposition (SVD)

$$A = U \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} V^\top \tag{6}$$

$$A^\top A = V\Sigma^2 V^{-1} \tag{7}$$

$$A^\top A\mathbf{v}_i = \lambda_i \mathbf{v}_i \qquad \lambda_i = \sigma_i^2 \tag{8}$$

$$AV = U \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} \tag{9}$$

$$U^+ = \Sigma^{-1} A V^+ \tag{10}$$

Numerical example

Find singular value decomposition

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 2 & 4 & 6 \end{bmatrix}$$