ECE 417/598: Singular Value Decompsition

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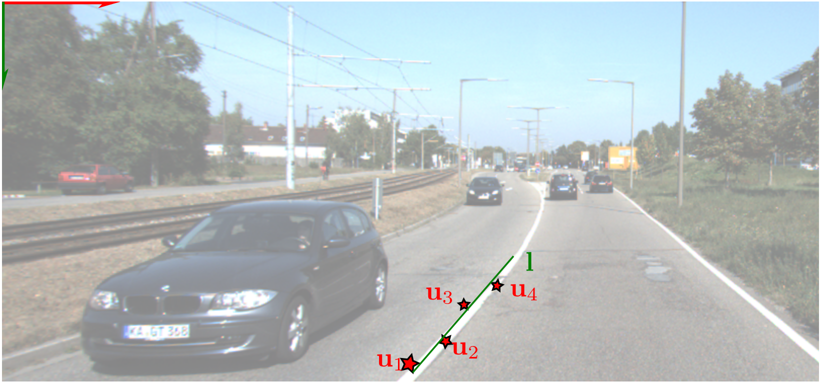
March 7, 2022



$$\underline{\mathbf{u}}_1 = [100, 98, 1]^\top$$
$$\underline{\mathbf{u}}_2 = [105, 95, 1]^\top$$
$$\underline{\mathbf{u}}_3 = [107, 90, 1]^\top$$
$$\underline{\mathbf{u}}_4 = [110, 85, 1]^\top$$

Find the line $\mathbf{l}$ such that it is the "closest line" passing through $\mathbf{u}_1, \ldots, \mathbf{u}_4$.

$$U = \begin{bmatrix} \mathbf{u}_1^\top \\ \mathbf{u}_2^\top \\ \mathbf{u}_3^\top \\ \mathbf{u}_4^\top \end{bmatrix}$$

We want to solve for $\mathbf{l}$ such that

$$U\mathbf{l} = 0$$

.

If the eigenvectors $x_1, \ldots, x_k$ correspond to different eigenvalues $\lambda_1, \ldots, \lambda_k$ then those eigenvectors are linearly independent.

$k=2$ · $\lambda_1 \neq \lambda_2$ $\Rightarrow$ $\underline{x}_1 = c_1 \underline{x}_2$

$\Rightarrow$ $c_1 \underline{x}_1 \neq c_2 x_2$

$c_1 \underline{x}_1 + c_2 \underline{x}_2 = 0$ — ①

$c_1, c_2 \in \mathbb{R}$

multiply bothsides with A

$c_1 A \underline{x}_1 + c_2 A \underline{x}_2 = 0$

$\Rightarrow$ $c_1 \lambda_1 \underline{x}_1 + c_2 \lambda_2 \underline{x}_2 = 0$ — ②

② $- \lambda_2$ ①

$\Rightarrow$ $c_1 \lambda_1 \underline{x}_1 - c_1 \lambda_2 \underline{x}_1 = 0$

$\Rightarrow$ $c_1 (\lambda_1 - \lambda_2) \underline{x}_1 = 0$ $\Rightarrow$

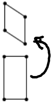
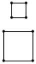
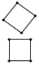
$\Rightarrow$ $c_1 \neq 0$ or $\underline{x}_1 = 0$

$\Rightarrow$ $c_1 \underline{x}_1 + c_2 \underline{x}_2 = 0 \Rightarrow c_2 = 0$

$c_1 = 0, \; c_2 = 0$

$\underline{x}_1, \underline{x}_2$ are linearly independent

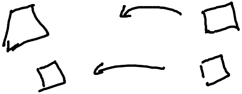
Hierarchy of transforms

Group	Matrix	Distortion	Invariant properties
Projective 8 dof	$\lambda \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix}$		Concurrency, collinearity, **order of contact**: intersection (1 pt contact); tangency (2 pt contact); inflections (3 pt contact with line); tangent discontinuities and cusps. cross ratio (ratio of ratio of lengths).
Affine 6 dof	$\begin{bmatrix} a_{11} & a_{12} & t_x \\ a_{21} & a_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$ 2, 2		Parallelism, ratio of areas, ratio of lengths on collinear or parallel lines (e.g. midpoints), linear combinations of vectors (e.g. centroids). The line at infinity, $\mathbf{l}_\infty$.
Similarity 4 dof	$s=2$ $\begin{bmatrix} sr_{11} & sr_{12} & t_x \\ sr_{21} & sr_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$		Ratio of lengths, angle. The circular points, **I**, **J** (see section 2.7.3).
Euclidean 3 dof	R, t $\begin{bmatrix} r_{11} & r_{12} & t_x \\ r_{21} & r_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$		Length, area

R=1 DOF 2 DOF

$\underline{y} = sR\underline{x} + t$

$\underline{x} = \underline{A}\,\underline{y}$



Conjugate (or Hermitian) Transpose

$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ $\quad A^T = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$ $\quad A \in \mathbb{R}^{3\times 3}$

$A \in \mathbb{C}^{m\times n}$ $\quad A = \underline{P} + i\underline{Q} = \begin{bmatrix} a_{11} + ib_{11} & \cdots & a_{1n} + ib_{1n} \\ a_{m1} + ib_{m1} & \cdots & a_{mn} + ib_{mn} \end{bmatrix}$

$\overline{a+ib} = a - ib$

$A^H = \bar{A}^T = P^T - iQ^T$

Hermitian or Symmetric matrices

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix} \Rightarrow \quad A = A^T \qquad A \in \mathbb{R}^{3\times 3}$$

$A \in \mathbb{C}^{n\times n}$ $\qquad A^H = A \quad \Rightarrow$ A is called a Hermitian matrix

$$A = \begin{bmatrix} 1+i & 2+i \\ 2-i & 3+i \end{bmatrix}$$

Property 1: If $A = A^H$, then for all complex vectors $x \in \mathbb{C}^n$, the number $\underbrace{x^H A x}_{\text{quadratic form}}$ is real.

$x^H A x \in \mathbb{R}$

$\underline{x}^H \in \mathbb{C}^{1 \times n} \quad A \in \mathbb{C}^{n \times n} \quad \underline{x} \in \mathbb{C}^{n \times 1}$

$a + ib = \underline{x}^H A \underline{x}$

$(a+ib)^H = (\underline{x}^H A \underline{x})^H$

$(y^T B z)^T \Rightarrow z^T B^T y$

$a - ib = \underline{x}^H A^H \underline{x}$

$a - ib = \underline{x}^H A \underline{x} = a + ib \quad \Rightarrow \quad b = 0$

Property 2: Every eigenvalue of a Hermitian matrix is real.

if $A = A^H$

$A^H \rightarrow A = A^H$

A Hermitian

A is a Hermitian matrix

$A\underline{x} = \lambda \underline{x}$ $\quad \lambda_1, \dots \lambda_n \in \mathbb{R}$

$A\underline{x}_i = \lambda_i \underline{x}_i$

multiply by x_i^H

$\underline{x}_i^H A \underline{x}_i = \lambda_i \, x_i^H x_i$

$\boxed{x_i^H x_i = \underline{x}_i^H I \underline{x}_i}$

$\Rightarrow \lambda_i = \dfrac{\underline{x}_i^H A \underline{x}_i \rightarrow \mathbb{R}}{x_i^H x_i \rightarrow \mathbb{R}}$

Property 3: The eigenvectors of a real symmetric matrix or a Hermitian matrix, if they come from different eigenvalues, are orthogonal to one another.
If $A = A^T$, the diagonalizing matrix S can be an orthogonal matrix $S^{-1} = S^T$ if they come from different eigenvalues.

$A \underline{x}_i = \lambda_i \underline{x}_i$ for all $i \in 1, \dots, n$

if $\lambda_i \neq \lambda_j$ $x_i^H x_j = 0$ for all $i \neq j$

$A x_i = \lambda_i x_i$, $A x_j = \lambda_j x_j$

multiply x_j^H

$\underline{x}_j^H A \underline{x}_i = \underline{\lambda}_i x_j^H \underline{x}_i$

$(A^H x_j)^H x_i = \lambda_i x_j^H x_i$

$(A\underline{x}_j)^H x_i = \lambda_i x_j^H x_i \Rightarrow$

$(A^H x_j)^H = x_j^H A$

$(\lambda_j x_j)^H x_i = \lambda_i x_j^H x_i$

$$\Rightarrow \quad \lambda_j \underline{\underline{x}_j^H \underline{x}_i} = \lambda_i \underline{\underline{x}_j^H \underline{x}_i}$$

$$\Rightarrow \quad (\lambda_i - \lambda_j)(\underline{x}_j^H \underline{x}_i) = 0 \qquad \Rightarrow \quad x_j^H x_i = 0$$

Singular Value Decomposition (SVD)

$$A = U \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} V^\top \tag{1}$$

$$A^\top A = V\Sigma^2 V^{-1} \tag{2}$$

$$A^\top A\mathbf{v}_i = \lambda_i \mathbf{v}_i \qquad \lambda_i = \sigma_i^2 \tag{3}$$

$$AV = U \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} \tag{4}$$

$$U^+ = \Sigma^{-1} A V^+ \tag{5}$$

$A \in \mathbb{R}^{m \times n}$

SVD

$B = A^T A \qquad \in \mathbb{R}^{n \times n}$

EVD of B $\qquad B = V \Lambda V^{-1}$

$V^{-1} = V^T$

$A^T A = V \Lambda V^T$

where,

Λ = diagonal matrix of eigen values

and V = matrix of eigen vector

$\lambda_i \ldots \lambda_n$ of $A^T A > 0$
or $= 0$

$V = [\underline{v_1} \; \ldots \; \underline{v_n}]$

$(A^T A) \underline{v_i} = \lambda_i \underline{v_i}$

Multiply with v_i^T on both sides

$v_i^T A^T A v_i^T = \lambda_i v_i^T v_i$

$$\lambda_i = \frac{v_i^T A^T A v_i}{v_i^T v_i} = \frac{\|A v_i\|^2 \geq 0}{\|v_i\|^2 > 0}$$

$\lambda_i \geq 0$

Singular values of A $\quad \sigma_i = \sqrt{\lambda_i}$ $\quad$ λ_i are eigen values of $A^T A$

SVD of $A = U \Sigma V^T$

singular value

$$\Sigma = \begin{bmatrix} \sigma_1 & & 0 \\ & \ddots & \\ 0 & & \sigma_n \end{bmatrix}$$

$$V^T = \begin{bmatrix} \underline{v}_1^T \\ \vdots \\ \underline{v}_n \end{bmatrix}$$

eigen vector of $A^T A$

right singular vector

U = left singular vector

$$u_i = \frac{A v_i}{\sigma_i}$$

$$AV = U\Sigma$$

$$U = AV\Sigma^{-1}$$

left singular vector $\sigma_i \neq 0$

diagonal matrix singular

$$A = \underbrace{U}\Sigma\underbrace{V^T}$$

orthonormal

Null space

$$A\underline{x} = 0$$

$$U\Sigma V^T \underline{x} = 0$$

$$\Sigma = \begin{bmatrix} \sigma_1 & 0 & & 0 \\ 0 & \ddots\ \sigma_r & & 0 \\ & & 0 & \\ 0 & & & \ddots\ 0 \end{bmatrix}$$

$$U\Sigma \begin{bmatrix} \underline{v}_1^T \\ \vdots \\ v_r^T \\ v_{r+1}^T \\ \vdots \\ v_n \end{bmatrix} \underline{x} = 0$$

$$N(A) = \underbrace{[v_{r+1} \cdots v_n]}_{(n-r)}$$

Numerical example

Find singular value decomposition

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 2 & 4 & 6 \end{bmatrix}$$