ECE 417/598: Review Homework 4

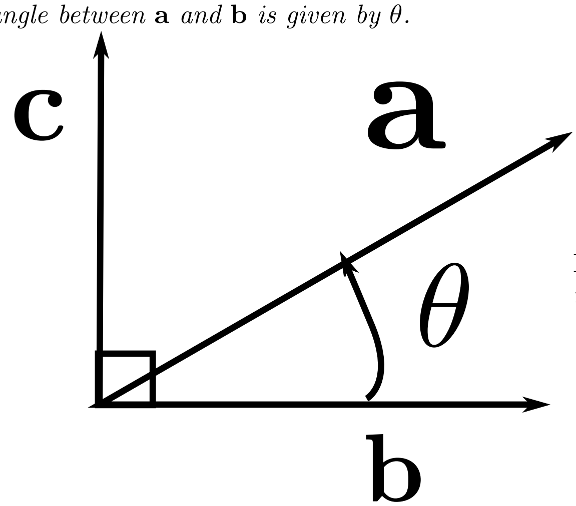
Max marks: 100 marks

Due on March 10th, 2021, midnight, 11:59 PM.

All notes so far are linked here.

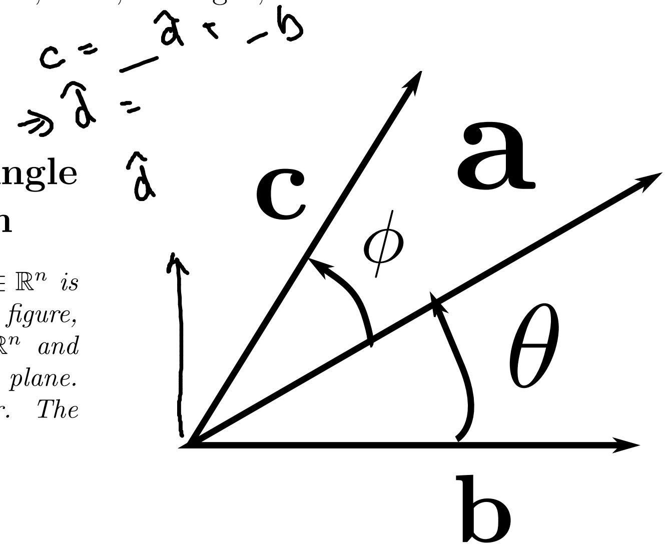
1 Trignometry and triangle laws of vector addition

Problem 1 *The magnitude of vector $\mathbf{a} \in \mathbb{R}^n$ is given to be $\|\mathbf{a}\| = \alpha$. Using the following figure, write $\mathbf{a}$ in terms of α, θ, vector $\mathbf{b} \in \mathbb{R}^n$ and $\mathbf{c} \in \mathbb{R}^n$. All three vectors lie in the same plane. $\mathbf{b}$ and $\mathbf{c}$ are perpendicular to each other. The angle between $\mathbf{a}$ and $\mathbf{b}$ is given by θ.*



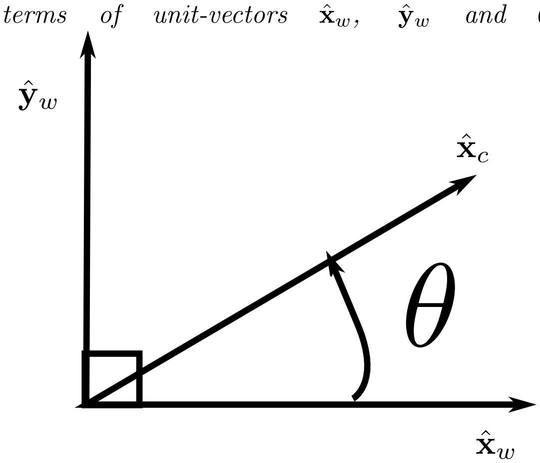
(5 min, 5 marks)

Problem 2 *The magnitude of vector $\mathbf{a} \in \mathbb{R}^n$ is given to be $\|\mathbf{a}\| = \alpha$. Using the following figure, write $\mathbf{a}$ in terms of α, θ, ϕ vector $\mathbf{b} \in \mathbb{R}^n$ and $\mathbf{c} \in \mathbb{R}^n$. All three vectors lie in the same plane. The angle between $\mathbf{a}$ and $\mathbf{b}$ is given by θ. The angle between $\mathbf{a}$ and $\mathbf{c}$ is given by ϕ. Assume $\theta+\phi \neq 0$. When $\theta+\phi = \pi/2$, is the solution is same as Problem 1? (Hint: You can convert this to Problem 1, by drawing a unit-vector perpendicular to $\mathbf{b}$. Call it $\hat{\mathbf{d}}$. First write $\hat{\mathbf{d}}$ in terms of $\mathbf{c}$ and others knowns and then write $\mathbf{a}$ in terms of $\hat{\mathbf{d}}$ and other knowns. You might want to use trignometric identities. The simplest form is not required.).*



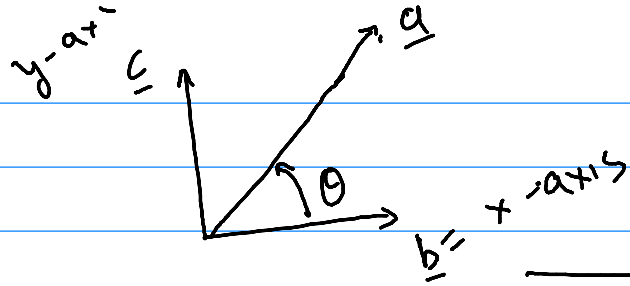
(10 min, 10 marks)

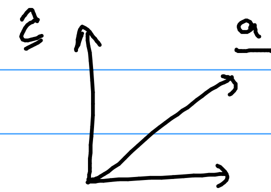
Problem 3 *Find unit-vector $\hat{\mathbf{x}}_c$ in terms of unit-vectors $\hat{\mathbf{x}}_w$, $\hat{\mathbf{y}}_w$ and θ.*



(5 min, 5 marks)

Problem 4 *Find unit-vector $\hat{\mathbf{y}}_c$ in terms of unit-vectors $\hat{\mathbf{x}}_w$, $\hat{\mathbf{y}}_w$ and θ.*





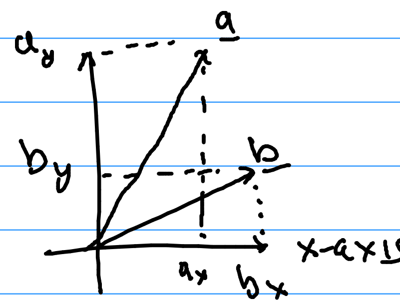
$$\hat{b} = \frac{b}{\|b\|}$$

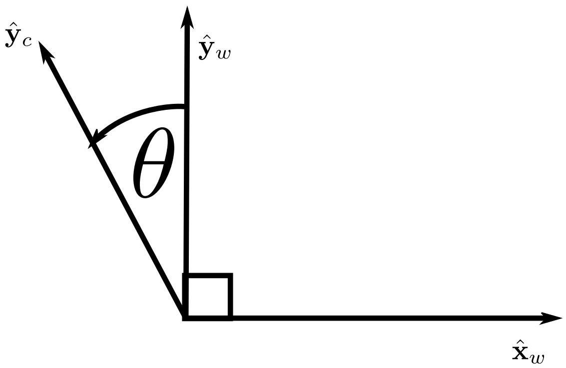
$$\boxed{a = R(\theta)\, b}$$

$$a = \alpha \frac{b}{\|b\|} \cos\theta + \frac{\alpha c}{\|c\|} \sin\theta$$

$$R(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

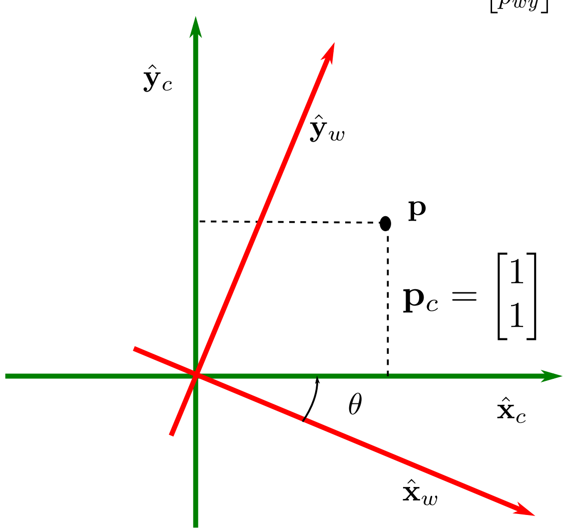
$$b = \begin{bmatrix} b_x \\ b_y \end{bmatrix}$$





(5 min, 5 marks)

Problem 5 *Let the coordinates of a vector* $\mathbf{p}$ *in terms of* $\hat{\mathbf{x}}_c$ *and* $\hat{\mathbf{y}}_c$ *be* $\mathbf{p}_c = \begin{bmatrix} p_{cx} \\ p_{cy} \end{bmatrix}$*, so that:* $\mathbf{p} = p_{cx}\hat{\mathbf{x}}_c + p_{cy}\hat{\mathbf{y}}_c$*. Using the results from Prob 3 and Prob 4, write* $\mathbf{p}$ *in terms of* $\hat{\mathbf{x}}_w$ *and* $\hat{\mathbf{y}}_w$*. Thus derive the formula for rotation matrix* $R(\theta)$ *that converts coordinates from* $\mathbf{p}_c$ *to* $\mathbf{p}_w = \begin{bmatrix} p_{wx} \\ p_{wy} \end{bmatrix}$.



(10 min, 10 marks)

Problem 6 *We know that* $\|\mathbf{v}_{\perp,rot}\| = \|\mathbf{v}_{\perp}\|$*. Write* $\mathbf{v}_{\perp,rot}$ *in terms of* $\mathbf{v}_{\perp}$*,* $\mathbf{w}$ *and* θ*.* $\mathbf{v}_{\perp}$ *and* $\mathbf{w}$ *are known to be orthogonal to each other.*

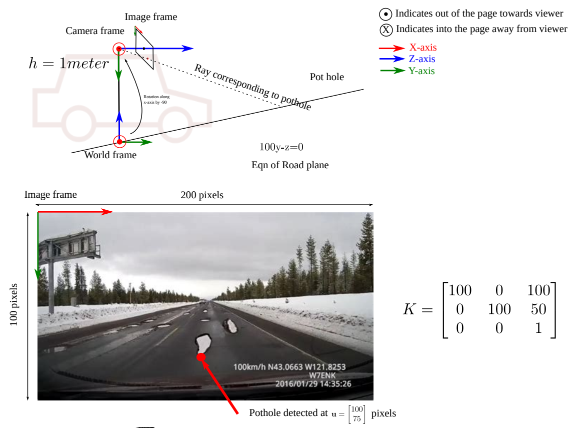
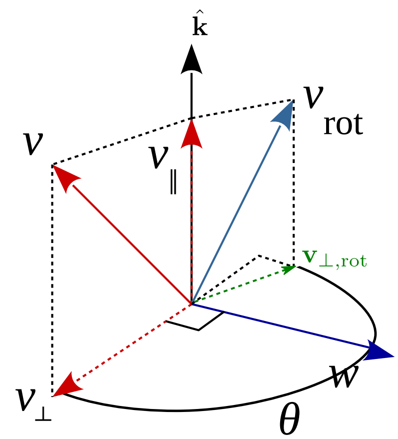


Figure 1: Point-plane triangulation



(5 min, 5 marks)

Problem 7 *In figure 1 find the 3D position of the pothole the World coordinate frame, in terms of* $h = 1$ *(the height of the camera), image-coordinates of the pothole* $\mathbf{u}$ *(provided in figure), camera matrix* K *(provided in figure). The Camera is mounted directly on top of the world frame, both of which are aligned to the gravity vector. The road is a perfect plane with a slope such that the equation of road plane in world-coordinate frame is given by* $100Y_w - Z_w = 0$ *and the pothole lies on the road plane. Provide the formula or pseudo-code for computing the pothole coordinates, and also substitute in the values. (20 min, 20 marks)*

Problem 8 *In figure 2 find the 3D representation of the lane the World coordinate frame, in terms of h (the height of the camera), image-representation of the line $\mathbf{l}$ (provided in figure), camera matrix K (provided in figure). Assume the lane to be a straight line. The Camera is mounted directly on top of the world frame, both of which are aligned to the gravity vector. The road is a perfect plane with a slope such that the equation of road plane in world-coordinate frame is given by $100Y_w - Z_w = 0$ and the lane lies on the road plane. Provide the formula or pseudo-code for computing the 3D representation of the lane, and also substitute in the values. (20 min, 20 marks)*

*Hint 0: **Equation of a plane in 3D.** Equation of a plane in 3D is given by $p_1X + p_2Y + p_3Z + p_4 = 0$. In matrix notation, you can write the equation plane as $\mathbf{p}_{1:3}^\top \mathbf{X} + p_4 = 0$, where $\mathbf{p}_{1:3} = [p_1, p_2, p_3]^\top$.*

*Hint 1: **3D Plane corresponding to the line in image-coordinates.** Let the equation of line in image-coordinates be $\mathbf{l}^\top \underline{\mathbf{u}} = 0$, where $\underline{\mathbf{u}} = \begin{bmatrix}\mathbf{u} \\ 1\end{bmatrix} \in \mathbb{P}^2$ are all the points on the line. By pinhole camera model, if $\mathbf{X}_c \in \mathbb{R}^3$ are the corresponding points in 3D, then the equation of corresponding plane is given by $\mathbf{l}^\top(K\mathbf{X}_c) = 0$ which can also be written as $(K^\top \mathbf{l})^\top \mathbf{X}_c = 0$. If we compare it to the equation of plane $\mathbf{p}_{1:3}^\top \mathbf{X} + p_4 = 0$, then $\mathbf{p}_{1:3} = K^\top \mathbf{l}$ and $p_4 = 0$.*

*Hint 2: **Intersection of two planes in 3D is a line.** Equation of a plane in 3D is given by $p_1X_w + p_2Y_w + p_3Z_w + p_4 = 0$. In matrix notation, you can write the equation of the plane as $\mathbf{p}_{1:3}^\top \mathbf{X}_w + p_4 = 0$, where $\mathbf{p}_{1:3} = [p_1, p_2, p_3]^\top$. Let's say you have two planes $\mathbf{p}_{1:3}^\top \mathbf{X}_w + p_4 = 0$ and $\mathbf{q}_{1:3}^\top \mathbf{X}_w + q_4 = 0$. Their intersection is a line whose parameteric form is given by (why ? you have all the knowledge required to derive this):*

$$\mathbf{X}_w = \lambda(\mathbf{p}_{1:3} \times \mathbf{q}_{1:3}) + \begin{bmatrix}\mathbf{p}_{1:3}^\top \\ \mathbf{q}_{1:3}^\top\end{bmatrix}^\dagger \begin{bmatrix}-p_4 \\ -q_4\end{bmatrix}, \qquad (1)$$

where $A^\dagger$ denotes the pseudo-inverse of a matrix (a fat matrix in this case) and $\lambda \in \mathbb{R}$ is the free parameter and $\times$ denotes the vector cross-product.

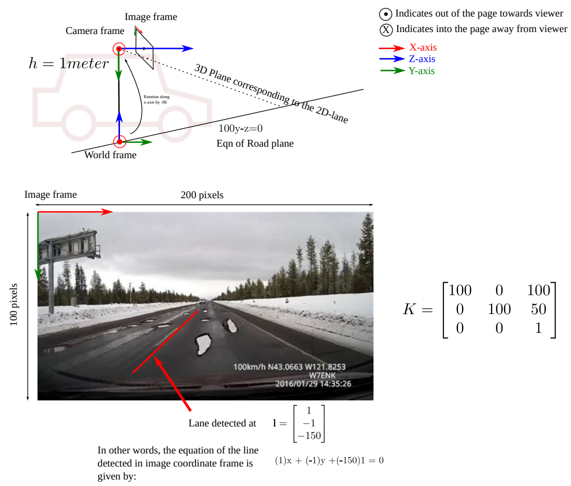


Figure 2: Line-plane triangulation

Problem 9 *You are a part of Tesla self-driving team. Team 1 provides you with lane-detection algorithms and their output. Team 2 provides you with detailed maps of road conditions. Your task is to write a function that solves problem 8 for arbitrary lanes detected by team 1 and for arbitrary plane provided by team 2. (Hint: Equation of a plane 3D is very similar to equation of line in 2D). What input representations of lane and plane would you ask for? Write a general algorithm or pseudo-code that solves problem 8. (30 min, 10 marks)*

$\underline{X}_c = \lambda K^{-1} \underline{u}$. $\qquad R = [\quad]$

$\underline{X}_w = R \underline{X}_c + t$

$\underline{X}_w = R \lambda K^{-1} \underline{u} + t \qquad \lambda \in \mathbb{R}$ unknown

$$\underline{X}_w = \lambda \begin{bmatrix} a \\ b \\ c \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ h \end{bmatrix}$$

$100 Y_w - Z_w = 0$

$100(\lambda b + 0) - (\lambda c + h) = 0$

$\lambda = ?$

Problem 8

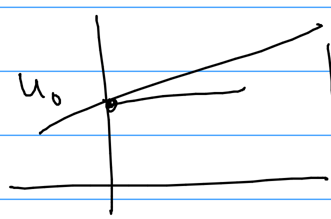
$[\ \underline{l}^T \underline{u} = \underline{u}^T \underline{l} = 0\ ]$.

implicit representation

Implicit representation

$(x - x_0)^2 + (y - y_0)^2 = r^2$

Parametric repr



$\underline{u} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix} t + u_0$

explicit representation

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{bmatrix} r\cos(t) + x_0 \\ r\sin(t) + y_0 \end{bmatrix}$$

$\underline{l}^T \underline{u} = 0 \qquad \underline{u} = \lambda K \underline{X}_c$

$l^T(\lambda K X_c) = 0$

$(K^T l)^T = l^T (K^T)^T = l^T K \qquad l^T K X_c = 0 \iff \underbrace{(K^T \underline{l})^T}_{1 \times 3} \underline{X}_c = 0$

$\underline{p}^T X_c = 0 \qquad \underline{p}$

$ax + by + cz + d = 0$ | implicit eq

$\underline{X}_w = R \underline{X}_c + t$

multiply both sides R^T

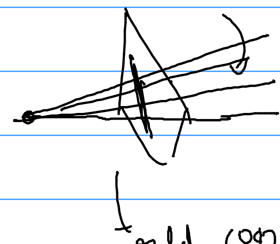
$R^T X_w = R^T R \underline{X}_c + R^T t$

Equation of plane in 3D in parametric

$\underline{X}_c = \alpha \underline{P}_1 + \beta \underline{P}_2$

$\underline{x}_c = R^T \underline{x}_w - R^T t$

Plane in 3D



$\underline{p}^T x_c = 0 \quad | \; \underline{p} = K^T \underline{l}$

$\Rightarrow p^T (R^T x_w - R^T \underline{t}) = 0$

$\Rightarrow p^T R^T \underline{x}_w - p^T R^T \underline{t} = 0$

① $\Rightarrow \underbrace{(Rp)^T}_{1\times 3} \underline{x}_w - \underbrace{p^T R^T t}_{1\times 1} = 0$

the equation of the plane in world coordinate

scs

$ax + by + cz + d = 0$

$ax + by + cz + d = 0$

② $\underline{q}^T x_w + q_4 = 0$

$100 Y_w - Z_w = 0 \quad | \; q = \begin{bmatrix} 0 \\ 100 \\ -1 \end{bmatrix}, \; q_4 = 0$

implicit repr of line

$$\begin{cases} \underline{p}^T \underline{x}_w + p_4 = 0 \\ \underline{q}^T \underline{x}_w + q_4 = 0 \end{cases}$$

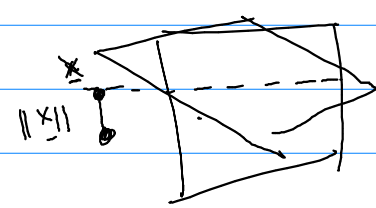
parametric of line in 3D

$\underline{x} = \underline{d}\lambda + \underline{x}_0$

↑ free parameter

$$\underbrace{\begin{bmatrix} p^T \\ q^T \end{bmatrix}}_{A} \underline{x}_w = \underbrace{\begin{bmatrix} -p_4 \\ -q_4 \end{bmatrix}}_{b}$$

(2 × 3)



$\underline{A}\,\underline{x}_w = \underline{b}$

$A \in \mathbb{R}^{2\times 3}$

$b \in \mathbb{R}^{2\times 1}$

a point on the line

$$\begin{cases} \min_x \; \|x\|^2 \\ \text{s.t.} \; Ax = b \end{cases}$$

$X_{w,0} = A^{\dagger} \underline{b}$

$X_{w,0} = A^T (AA^T)^{-1} \underline{b}$

$A(x_w - x_{w,0}) = Ax_w - (AA^T(AA^T)^{-1}\underline{b})$

$= \underline{b} - \underline{b} = 0$

$x_w - x_{w,0} \in N(A)$ $\qquad$ $A(x_w - x_{w,0}) = \underline{0}$

$N(A) = \lambda(\underline{p} \times \underline{q})$ $\qquad$ $A = \begin{bmatrix} \underline{p}^T \\ \underline{q}^T \end{bmatrix}$

$\underline{x_w} = \lambda(\underline{p \times q}) + \underline{x_{w,0}}$

↑ free parameter; $\underline{p \times q}$: direct; ↑ $x_{w,0}$: point on the line