ECE 417/598: Plane to points and DLT

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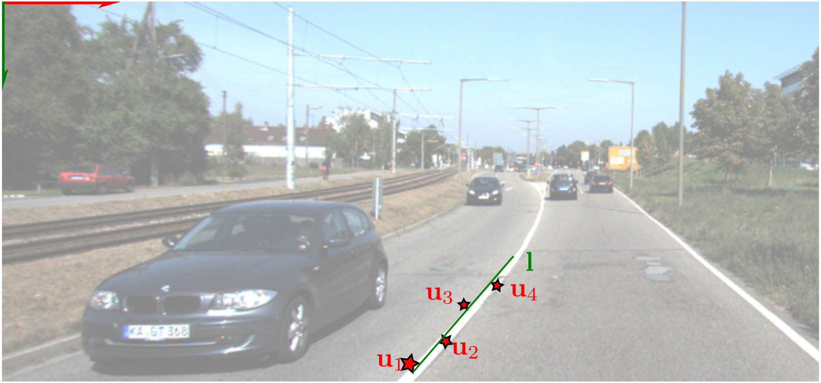


$$\underline{\mathbf{u}}_1 = [100, 98, 1]^\top$$
$$\underline{\mathbf{u}}_2 = [105, 95, 1]^\top$$
$$\underline{\mathbf{u}}_3 = [107, 90, 1]^\top$$
$$\underline{\mathbf{u}}_4 = [110, 85, 1]^\top$$

Find the line $\mathbf{l}$ such that it is the "closest line" passing through $\mathbf{u}_1, \ldots, \mathbf{u}_4$.

$$A = \begin{bmatrix} \mathbf{u}_1^\top \\ \mathbf{u}_2^\top \\ \mathbf{u}_3^\top \\ \mathbf{u}_4^\top \end{bmatrix}$$

$\underline{u}^T \cdot \underline{\ell} = 0$

We want to solve for $\mathbf{l}$ such that

$$A\mathbf{l} = 0$$

$\ell \in \mathcal{N}(A)$

Singular Value Decomposition (SVD)

$$A = U \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} V^\top$$

$$\boxed{A^\top A} = V \Sigma^2 V^{-1}$$

$$A^\top A \mathbf{v}_i = \lambda_i \mathbf{v}_i \qquad \lambda_i = \sigma_i^2, \Sigma = \mathrm{diag}([\sigma_1, \ldots, \sigma_r])$$

$$\boxed{\mathbf{u}_i = \frac{A\mathbf{v}_i}{\sigma_i}}$$

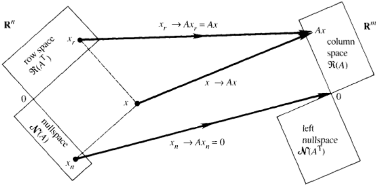
$$U = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \ldots & \mathbf{u}_m \end{bmatrix}$$

If $A \in \mathbb{R}^{m\times n}$ and the rank of A is r, then

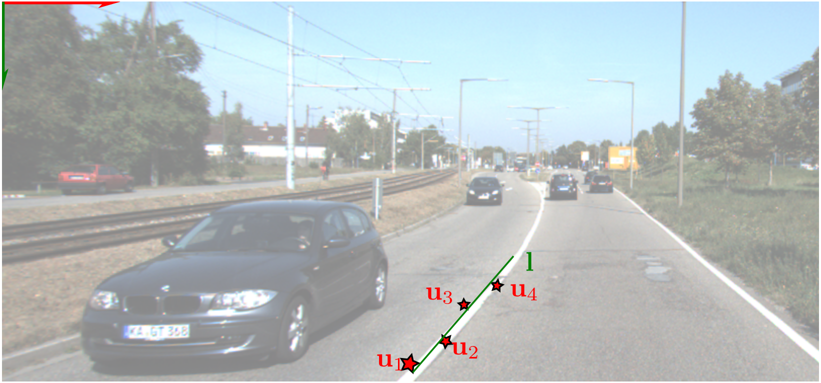
$$A = \begin{bmatrix} U_{m\times r} & U_{m\times(m-r),\perp} \end{bmatrix} \begin{bmatrix} \Sigma_{r\times r} & 0_{r\times(n-r)} \\ 0_{(m-r)\times r} & 0_{(m-r)\times(n-r)} \end{bmatrix} \begin{bmatrix} V_{n\times r}^{\top} \\ V_{n\times(n-r),\perp}^{\top} \end{bmatrix}$$

$$A = U_{m\times r}\Sigma_{r\times r}V_{n\times r}^{\top} + 0 * U_{m\times(m-r),\perp}V_{n\times(n-r),\perp}^{\top}$$

$V_{n\times(n-r)\perp}$



$$
\begin{aligned}
\mathcal{N}(A) &= V_{n\times(n-r),\perp} \\
\mathcal{R}(A) &= U_{m\times r} \\
\mathcal{N}(A^\top) &= U_{m\times(m-r),\perp} \\
\mathcal{R}(A^\top) &= V_{n\times r}
\end{aligned}
$$



We want to solve for $\mathbf{l}$ such that

$$A\mathbf{l} = 0$$

$A = \begin{bmatrix} u_1^T \\ u_2^T \\ \vdots \\ u_n^T \end{bmatrix}$

A is $m \times 3$ and has rank 2. Solution

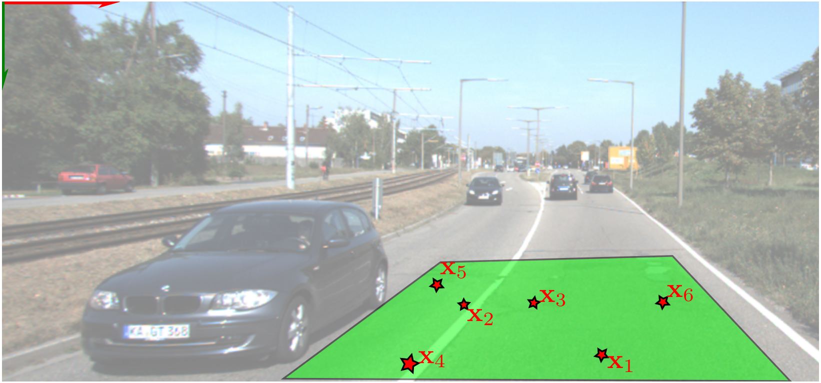
$$U\Sigma V^\top = A$$

$$V = [\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3]$$

$$\boxed{\mathbf{l} = \mathbf{v}_3}$$

$\boxed{A}x = 0$

$x \in N(A)$



$\underline{x}_2^\top \underline{p} = 0$ $\qquad$ $\underline{x}_i^\top \underline{p} = 0$ $\qquad$ $ax + by + cz + d = 0$

$\underline{x}_3^\top p = 0$

$\underline{\mathbf{x}}_1 = [-2.3, 1.04, 3.2, 1]^\top \qquad \underline{\mathbf{x}}_2 = [-2.2, 1.02, 2.2, 1]^\top$

$x_1 = \begin{bmatrix} \\ \\ \end{bmatrix}$ $\qquad$ $\underline{\mathbf{x}}_3 = [-2.1, 1.01, 1.2, 1]^\top \qquad \underline{\mathbf{x}}_4 = [2.1, 1.04, 1.2, 1]^\top$

$\underline{\mathbf{x}}_5 = [2.2, 1.03, 3.2, 1]^\top \qquad \underline{\mathbf{x}}_6 = [2.3, 1.01, 4.2, 1]^\top$

$[a, b, c, d] \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = 0$

Find the equation of plane $\mathbf{p} = [p_1, p_2, p_3, p_4]^\top$ such all points lie on the plane.

$$A = \begin{bmatrix} \underline{x}_1 & \underline{x}_2 & \underline{x}_3 & \underline{x}_4 \end{bmatrix} \Leftrightarrow \quad A = \begin{bmatrix} \underline{x}_1^T \\ \underline{x}_2^T \\ \underline{x}_3^T \end{bmatrix}$$

$$\underline{x}_1^T \underline{p} = 0 \qquad \underline{p}^T \underline{x}_1 = 0$$

$$\underline{x}_2^T \underline{p} = 0 \qquad \underline{p}^T \underline{x}_2 = 0$$

$$\underline{x}_3^T \underline{p} = 0$$

$$\vdots$$

$$\underbrace{\begin{bmatrix} \underline{x}_1^T \\ \underline{x}_2^T \\ \vdots \\ x_{3n}^T \end{bmatrix}}_{A} \underline{p} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad \underline{p}^T \underbrace{\begin{bmatrix} \underline{x}_1 & \underline{x}_2 & \underline{x}_3 \cdots & x_4 \end{bmatrix}}_{B} = 0$$

$$p^T B = 0 \Leftrightarrow A^T \underline{p} = 0$$

$$B^T = A \qquad p \in N(B^T)$$

$$\boxed{A\underline{p} = 0}$$

$$p = N(A)$$

$$A = U \Sigma V^T \qquad \text{rank}(A) = ?$$

$$A = \begin{bmatrix} x_1^T \\ x_2^T \\ \vdots \\ x_6^T \end{bmatrix} \in \mathbb{R}^{6\times 4} \qquad \text{rank}(A) < 4$$

$$x_1 = [x_1, y_1, z_1, 1]$$

$$x_2 = [x_2, y_2, z_2, 1]$$

$$\boxed{\text{rank}(A) = 3}$$

$$ax + by + cz + d = 0$$

$$\frac{a}{d}x + \frac{b}{d}y + \frac{c}{d}z + 1 = 0$$

$$\underset{6\times 4}{A} = U \Sigma V^T = U_{6\times 6} \begin{array}{c c} & 4 \\ 6 & \begin{pmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \sigma_3 & \\ & & & 0.01 \\ & & & \\ & & & \end{pmatrix} \end{array} \begin{array}{c c} 4 & \\ \begin{bmatrix} v_1^T \\ v_2^T \\ v_3^T \\ v_4^T \end{bmatrix} & 4 \end{array}$$

$$\underline{p} = N(A) = \cdot \underline{v}_4$$

$$A_{6\times 4} = U_{6\times 6}\,\Sigma_{6\times 4}\,V^{T}_{4\times 4}$$

$$\text{rank}(A) = 3$$

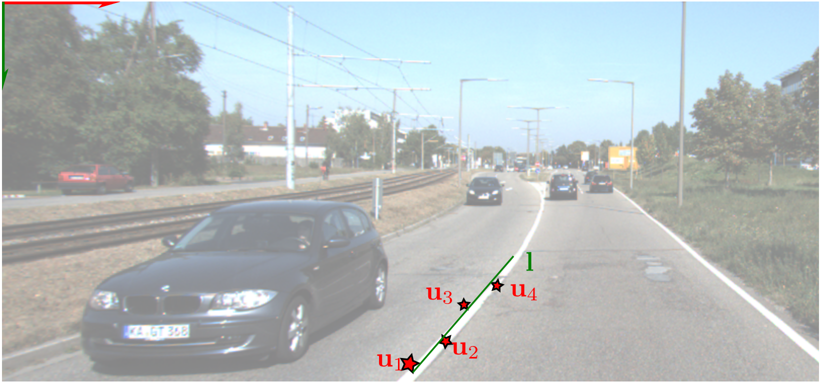
$$A = \begin{bmatrix} U_{6\times 3} & U'_{6\times 3} \end{bmatrix} \begin{array}{cc} \begin{bmatrix} \sigma_1 & 0 & 0 & 0 \\ 0 & \sigma_2 & 0 & 0 \\ 0 & 0 & \sigma_3 & 0 \\ 0 & 0 & 0 & 0.01 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} & \begin{array}{c} 4 \\ \\ \\ \\ 6 \end{array} \end{array} \begin{bmatrix} V^{T}_{4\times 3} \\ V'^{T}_{4\times 1} \end{bmatrix}$$

$$\boxed{A = U_{6\times 3}\begin{bmatrix} \sigma_1 & & 0 \\ & \sigma_2 & \\ 0 & & \sigma_3 \end{bmatrix} V^{T}_{4\times 3}} + U'_{6\times 3}\begin{bmatrix} 0.01 \\ 0 \\ 0 \end{bmatrix} V'^{T}_{4\times 1}$$

nullspace

$$N(A) = \underline{v}_4$$

$$A\underline{p} = 0 \Rightarrow \underline{p} = \underline{v}_4$$



Line in 3D?

$$\begin{cases} \underline{\mathbf{x}}_1 = [100, 98, 45, 1]^\top \\ \underline{\mathbf{x}}_2 = [105, 95, 46, 1]^\top \\ \underline{\mathbf{x}}_3 = [107, 90, 47, 1]^\top \\ \underline{\mathbf{x}}_4 = [110, 85, 43, 1]^\top \end{cases}$$

Find the 3D line such that it is the "closest line" passing through $\mathbf{x}_1, \ldots, \mathbf{x}_4 \in \mathbf{P}^3$.

Implicit equation of a line in 3D

$$\left[\begin{array}{l} \underline{p}_1^T \underline{x} = 0 \\ \underline{p}_2^T \underline{x} = 0 \end{array}\right] \qquad p_1 = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \qquad p_2 = \begin{pmatrix} e \\ f \\ g \\ h \end{pmatrix}$$

$$\underline{x}_1^T \underline{p}_1 = 0 \qquad \underline{x}_1^T \underline{p}_2 = 0$$

$$\underline{x}_2^T \underline{p}_1 = 0 \qquad \underline{x}_2^T \underline{p}_2 = 0$$

$$\underline{x}_3^T \underline{p}_1 = 0 \qquad \vdots$$

$$\vdots$$

$$A = \begin{bmatrix} \underline{x}_1^T \\ \underline{x}_2^T \\ \vdots \end{bmatrix}_{6\times 4}$$

$$\left.\begin{array}{l} A\,\underline{p}_1 = 0 \\ A\,\underline{p}_2 = 0 \end{array}\right] \qquad A\begin{bmatrix} \underline{p}_1 & \underline{p}_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\text{rank}(A) = 3$$

$$\text{rank}(A) = 2$$

$$\begin{array}{cccc} & 6 \quad U & \Sigma & V^T \end{array}$$

$$A = \underset{6}{\begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}} \; 6\begin{bmatrix} \sigma_1 & & 0 & 0 \\ & \sigma_2 & 0 & \\ & & 0.01 & 0 \\ & 0 & & 0.01 \\ & & & \end{bmatrix}_{\xleftrightarrow{\;4\;}} \begin{bmatrix} \underline{v_1^T} \\ v_2^T \\ v_3^T \\ v_4^T \end{bmatrix}$$

$Ax = 0 \quad \Rightarrow \quad x = \lambda_3 v_3 + \lambda_4 v_4 \qquad \lambda_3, \lambda_4 \in \mathbb{R}$

$\mathcal{N}(A)$

$A v_3 = 0 \qquad A v_4 = 0$

$A(1 v_3 + 2 v_4) = 0$

$A v_3 = 0$

$A v_4 = 0$

Eq of line

$\underline{v_3}^T \underline{x} = 0$

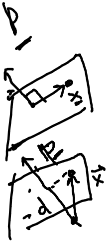
$\underline{v_4}^T \underline{x} = 0$

Implicit and parameteric equations of lines and plane

① Plane in 3D

Implicit : $ax + by + cz + d = 0$

$$\underbrace{\begin{bmatrix} a & b & c & d \end{bmatrix}}_{\underline{p}^T} \underbrace{\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}}_{\underline{x}} = 0$$



$$\underline{p}^T \underline{x} = 0$$

$$\underbrace{\begin{bmatrix} a & b & c \end{bmatrix}} \begin{bmatrix} x \\ y \\ z \end{bmatrix} + d = 0$$

$$p_{1:3}^T \vec{x} + d = 0 \Rightarrow p_{1:3}^T \underline{\vec{x}} = -d$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{Bmatrix} r\cos\theta \\ r\sin\theta \end{Bmatrix}$$

Parametric

$$\underline{x} = \{ \quad t_1, \; t_2 \}$$

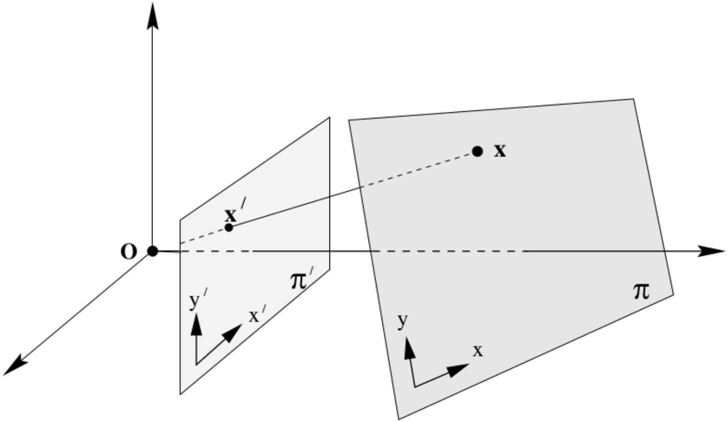
$$\underline{p}^T \underline{x} = 0$$

$$A \underline{x} = 0$$

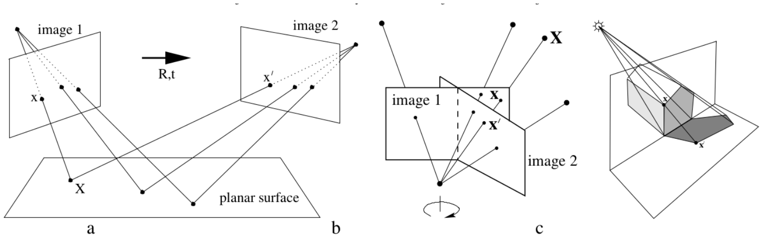
$$\underline{x} = \mathcal{N}(p^T) \qquad 4\times 1$$

$$\boxed{\underline{x} = \left[\lambda_2 \underline{v}_2 + \lambda_3 \underline{v}_3 + \lambda_4 \underline{v}_4 \right]}$$

Homography

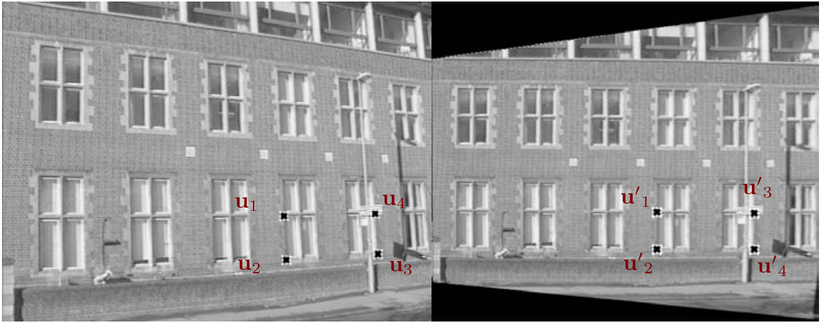


Examples of Homography




Top view
Graphic
Look! Safe to move?
Settings

Computing Homography



$$\underline{\mathbf{u}}_1 = [100, 98, 1]^\top \qquad \underline{\mathbf{u}}_2 = [102, 95, 1]^\top$$

$$\underline{\mathbf{u}}_3 = [107, 90, 1]^\top \qquad \underline{\mathbf{u}}_4 = [110, 85, 1]^\top$$

$$\underline{\mathbf{u}}'_1 = [100, 98, 1]^\top \qquad \underline{\mathbf{u}}'_2 = [102, 95, 1]^\top$$

$$\underline{\mathbf{u}}'_3 = [107, 98, 1]^\top \qquad \underline{\mathbf{u}}'_4 = [110, 85, 1]^\top$$

Find H such that $\underline{\mathbf{u}}' = H\underline{\mathbf{u}}$ for any point on one image to another image.

2D homography

Given a set of points $\underline{\mathbf{u}}_i \in \mathbb{P}^2$ and a corresponding set of points $\underline{\mathbf{u}}'_i \in \mathbb{P}^2$, compute the projective transformation that takes each $\underline{\mathbf{u}}_i$ to $\underline{\mathbf{u}}'_i$. In a practical situation, the points $\underline{\mathbf{u}}_i$ and $\underline{\mathbf{u}}'_i$ are points in two images (or the same image), each image being considered as a projective plane $\mathbb{P}^2$.

Solving for Homography

3D to 2D camera projection matrix estimation

Given a set of points $\mathbf{X}_i$ in 3D space, and a set of corresponding points $\mathbf{x}_i$ in an image, find the 3D to 2D projective $\mathbf{P}$ mapping that maps $\mathbf{X}_i$ to $\mathbf{x}_i = \mathbf{P}\mathbf{X}_i$.