ECE 417/598: K,R,t from P matrix

Instructor: Vikas Dhiman

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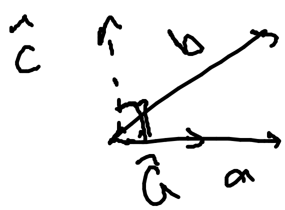
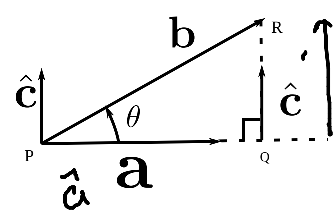
(1) Student name: Student email:

Rules

1. Let unit vectors of $\mathbf{a}$ be denoted by $\hat{\mathbf{a}} = \frac{\mathbf{a}}{\|\mathbf{a}\|}$.
2. The projection of $\mathbf{b}$ on $\mathbf{a}$ is $\overrightarrow{PQ}$. The magnitude of the projection is given by dot product,
$$|\overrightarrow{PQ}| = \mathbf{b}^\top \hat{\mathbf{a}} = \hat{\mathbf{a}}^\top \mathbf{b} = \|\hat{\mathbf{a}}\|\|\mathbf{b}\| \cos(\theta) = \|\mathbf{b}\| \cos(\theta)$$
3. Since $\overrightarrow{PQ}$ is in the direction of $\hat{\mathbf{a}}$, the vector $\overrightarrow{PQ}$ is given by,
$$\overrightarrow{PQ} = |\overrightarrow{PQ}|\hat{\mathbf{a}} = (\mathbf{b}^\top \hat{\mathbf{a}})\hat{\mathbf{a}}$$
4. Similarly the projection of $\mathbf{b}$ on $\hat{\mathbf{c}}$ is $\overrightarrow{QR}$
$$|\overrightarrow{QR}| = \mathbf{b}^\top \hat{\mathbf{c}} = \hat{\mathbf{c}}^\top \mathbf{b} = \|\hat{\mathbf{a}}\|\|\mathbf{b}\| \cos(\frac{\pi}{2} - \theta) = \|\mathbf{b}\| \cos(\frac{\pi}{2} - \theta)$$
$$\overrightarrow{QR} = (\mathbf{b}^\top \hat{\mathbf{c}})\hat{\mathbf{c}}$$
5. By triangle law $\mathbf{b} = \overrightarrow{PQ} + \overrightarrow{QR}$, or
$$\mathbf{b} = (\mathbf{b}^\top \hat{\mathbf{a}})\hat{\mathbf{a}} + (\mathbf{b}^\top \hat{\mathbf{c}})\hat{\mathbf{c}}$$



$\hat{a} = \frac{a}{\|a\|}$



Problem 1

We want to find a pair of orthonormal vectors in the same plane as $\mathbf{a}$ *and* $\mathbf{b}$*. First vector is* $\hat{\mathbf{a}}$*. What is the second vector? (Call it* $\hat{\mathbf{c}}$ *and find it in terms of* $\mathbf{a}$ *and* $\mathbf{b}$*.)*

$\hat{a}^T \hat{c} = 0$

$\hat{c}^T \hat{a} = 0$

$$\underbrace{(b^T \hat{c})}_{r} \hat{c} = \underline{b} - (\underline{b}^T \underline{\hat{a}}) \underline{\hat{a}}$$

$$\hat{c} = \frac{\underline{b} - (b^T \hat{a})\hat{a}}{\| b - (b^T \hat{a})\hat{a} \|}$$

Problem 2 *Express the above relationship in terms of matrix vector multiplication so that the matrix* $M = \begin{bmatrix} \mathbf{b}^\top \\ \mathbf{a}^\top \end{bmatrix}$ *can be written in terms of an upper triangular matrix and an orthonormal matrix.*

$$\hat{a} = \frac{a}{\|a\|}$$

$$b = (b^T\hat{a})\hat{a} + (b^T\hat{c})\hat{c}$$

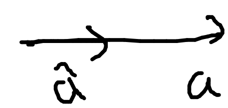
$$a = \|a\|\hat{a}$$

$$\not\Rightarrow \quad M = KR = \begin{bmatrix} \sim & \sim \\ 0 & \sim \end{bmatrix} \begin{bmatrix} \hat{c}^T \\ \hat{a}^T \end{bmatrix}$$

$$\begin{cases} b^T = (b^T\hat{a})\,\hat{a}^T + (b^T\hat{c})\,\hat{c}^T \\ a^T = \|a\|\,\hat{a}^T \end{cases}$$

$$\begin{bmatrix} b^T \\ a^T \end{bmatrix} = \underbrace{\begin{bmatrix} b^T\hat{c} & b^T\hat{a} \\ 0 & \|a\| \end{bmatrix}}_{2\times 2} \begin{bmatrix} \hat{c}^T \\ \hat{a}^T \end{bmatrix}$$

$$\|a\| = a^T\hat{a} \;?$$



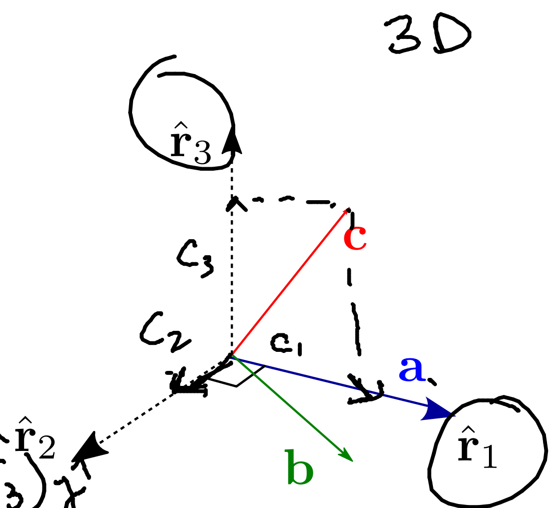
$$a^T\hat{a} = \|a\|\underbrace{\|\hat{a}\|}_{1}\underbrace{\cos 0}_{1} = \|a\|$$

$$M = \begin{bmatrix} b^T \\ a^T \end{bmatrix} = \underbrace{\begin{bmatrix} b^T\hat{c} & b^T\hat{a} \\ 0 & a^T\hat{a} \end{bmatrix}}_{\text{upper triangular}} \underbrace{\begin{bmatrix} \hat{c}^T \\ \hat{a}^T \end{bmatrix}}_{\text{orthonormal}}$$

Q-R factorization

GRAM SCHMIDT Process

Problem 3 *Repeat the process for 3 vectors,* **a**, **b** *and* **c**, *and then matrix* $M = \begin{bmatrix} \mathbf{c}^\top \\ \mathbf{b}^\top \\ \mathbf{a}^\top \end{bmatrix}$. *In other words, find* $\hat{\mathbf{r}}_1$, $\hat{\mathbf{r}}_2$, $\hat{\mathbf{r}}_3$ *and write them in (upper triangular matrix) (orthonormal matrix) factorization form, also known as QR factorization.*



$$c_1 = c^T \hat{r}_1$$
$$c_2 = c^T \hat{r}_2$$
$$c_3 = c^T \hat{r}_3$$
$$c = (c^T \hat{r}_1)\hat{r}_1 + (c^T \hat{r}_2)\hat{r}_2 + (c^T \hat{r}_3)\hat{r}_3$$

1. Write $\hat{\mathbf{r}}_1$ in terms of **a**.

$$\hat{r}_1 = \frac{a}{\|a\|} \Big\} \text{unit vecto}$$

$$a = \|a\| \hat{r}_1 = (a^T \hat{r}_1)\hat{r}_1 \quad (1)$$

2. Write $\hat{\mathbf{r}}_2$ in terms of **a**, **b** and $\hat{\mathbf{r}}_1$.

$$\hat{r}_2 = \frac{b - (b^T \hat{r}_1)\hat{r}_1}{\| b - (b^T \hat{r}_1)\hat{r}_1 \|}$$

$$b = (b^T \hat{r}_2)\hat{r}_2 + (b^T \hat{r}_1)\hat{r}_1 \quad (2)$$

3. Write $\hat{\mathbf{r}}_3$ in terms of **a**, **b**, **c**, $\hat{\mathbf{r}}_1$ and $\hat{\mathbf{r}}_2$.

$$\hat{r}_3 = \frac{c - (c^T \hat{r}_2)\hat{r}_2 - (c^T \hat{r}_1)\hat{r}_1}{\| c - (c^T \hat{r}_2)\hat{r}_2 - (c^T \hat{r}_1)\hat{r}_1 \|}$$

$$c = (c^T \hat{r}_1)\hat{r}_1 + (c^T \hat{r}_2)\hat{r}_2 + (c^T \hat{r}_3)\hat{r}_3 \quad (3)$$

$$a = (a^T \hat{r}_1)\,\hat{r}_1$$

$$b = (b^T \hat{r}_2)\,\hat{r}_2 + (b^T \hat{r}_1)\,\hat{r}_1$$

$$c = (c^T \hat{r}_1)\,\hat{r}_1 + (c^T \hat{r}_2)\,\hat{r}_2 + (c^T \hat{r}_3)\,\hat{r}_3$$

4. Write the above equations in matrix multiplication form.

$$\begin{bmatrix} c^T \\ b^T \\ a^T \end{bmatrix} = \begin{bmatrix} c^T\hat{r}_3 & c^T\hat{r}_2 & c^T r_1 \\ 0 & b^T r_2 & b^T r_1 \\ 0 & 0 & a^T r_1 \end{bmatrix} \begin{bmatrix} r_3^T \\ r_2^T \\ r_1^T \end{bmatrix}$$

Problem 4 *Assuming a QR factorization algorithm is given, find* $K, R, \mathbf{t}$ *from* $P \in \mathbb{R}^{3\times4}$ *matrix such that*

$$P = [\underbrace{KR}_{3\times3} \quad \underbrace{K\mathbf{t}}_{3\times1}] \quad 3\times4$$

and $K \in \mathbb{R}^{3\times3}$ *is an upper triangular matrix, and* $R \in \mathbb{R}^{3\times3}$ *is a rotation matrix (thus orthonormal) and* $\mathbf{t} \in \mathbb{R}^{3\times1}$ *is a translation vector.*

$$K = \begin{bmatrix} f_x & s & u_x \\ 0 & f_y & u_y \\ 0 & & 1 \end{bmatrix}$$

$$P = [M \quad Kt]$$

$$M = \begin{bmatrix} c^T \\ b^T \\ a^T \end{bmatrix}_{3\times3} \qquad M = KR$$

$$f_x = c^T \hat{r}_3$$

$$P.\mathrm{col}(3) = P(:,4) = Kt$$

$$t = K^{-1} P(:,4) = K.\mathrm{inverse}() * P.\mathrm{col}(3)$$