ECE 417/598: What can you find from two images

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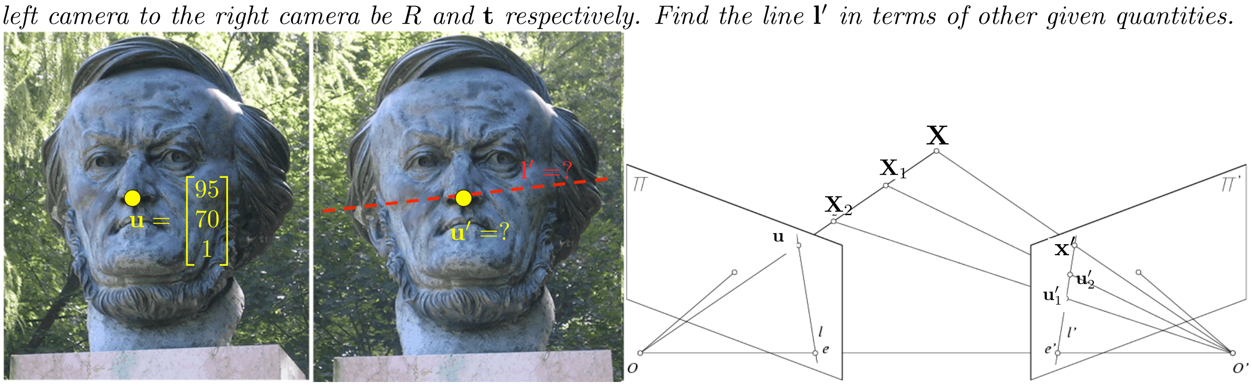
Things to recall

1. Projection of a 3D point to an image $\lambda \mathbf{u} = K\mathbf{X}$.
2. Transformation of coordinates $\mathbf{X}' = R\mathbf{X} + \mathbf{t}$.
3. A 2D line passing through two points $\mathbf{u}_1 \in \mathbb{P}^2$ and $\mathbf{u}_2 \in \mathbb{P}^2$ is given by cross product $\mathbf{l} = \mathbf{u}_1 \times \mathbf{u}_2$.
4. A cross product can be represented as a matrix vector product,

$$\mathbf{u}_1 \times \mathbf{u}_2 = \begin{bmatrix} 0 & -x_1 & y_1 \\ x_1 & 0 & -w_1 \\ -y_1 & w_1 & 0 \end{bmatrix} \mathbf{u}_2$$

   , where $\mathbf{u}_1 = [x_1, y_1, w_1]^\top$.
5. Cross product magnitude is given by $\|\mathbf{u}_1 \times \mathbf{u}_2\| = \|\mathbf{u}_1\|\|\mathbf{u}_2\| \sin(\theta)$ where θ s the angle between $\mathbf{u}_1$ and $\mathbf{u}_2$.

Problem 1 *When the depth corresponding to the point* $\mathbf{u}$ *in the left image is unknown, the possible pixels* ($\mathbf{u}'$) *on the right image that can correspond to the point form a line* $\mathbf{l}'$. *Let the intrinsic camera matrix of left camera be* K, *that of right camera be* K'. *Let the rotation and translation of the camera coordinates from the left camera to the right camera be* R *and* $\mathbf{t}$ *respectively. Find the line* $\mathbf{l}'$ *in terms of other given quantities.*



1. Find the equation of ray corresponding to the pothole in the left image. In other words, write the equation of the ray corresponding to the point $\mathbf{u}$, in the first camera frame.

$$\mathbf{X} = \lambda K^{-1}\mathbf{u}$$

2. Assume the rotation matrix R and translation vector $\mathbf{t}$, so that they transform any point in the left coordinate frame to a point in the right coordinate frame. ($\mathbf{X}' = R\mathbf{X} + \mathbf{t}$). Transform the equation of ray from the left coordinate frame to the right coordinate frame.

$$\mathbf{X}' = R\mathbf{X} + \mathbf{t}$$

3. Project the 3D point $\mathbf{X}'$ on the right camera, call it $\mathbf{u}'$.

$$\mathbf{u}' = \lambda' K' \mathbf{X}' = \lambda' K'(R\mathbf{X} + \mathbf{t}) = \lambda' K'(R(\lambda K^{-1}\mathbf{u}) + \mathbf{t}) = \lambda' \lambda K' R K^{-1}\mathbf{u} + \lambda' K' \mathbf{t}$$

4. The point on right image $\mathbf{e}'$ is the projection of origin $(0, 0)$ of left camera. Find $\mathbf{e}'$ (also called the

$$\mathbf{e}' = \lambda'_e K' R \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \lambda'_e K' \mathbf{t} = \lambda'_e K' \mathbf{t}$$

5. Find a line $\mathbf{l}'$ that passes through both $\mathbf{u}'$ and $\mathbf{e}'$, such that $\mathbf{l}'^\top \mathbf{u}' = 0$ and $\mathbf{l}'^\top \mathbf{e}' = 0$.

$$\mathbf{l}' = \mathbf{e}' \times \mathbf{u}' = \lambda'_e K' \mathbf{t} \times (\lambda' \lambda K' R K^{-1}\mathbf{u} + \lambda' K' \mathbf{t}) = \lambda'_e K' \mathbf{t} \times \lambda' \lambda K' R K^{-1}\mathbf{u} + \underbrace{\lambda'_e K' \mathbf{t} \times \lambda' K' \mathbf{t}}_{=\mathbf{0}_{3\times 1}}$$

$$\mathbf{l}' = K' \mathbf{t} \times K' R K^{-1}\mathbf{u}$$

6. Define Fundamental matrix F such that $\mathbf{u}'^\top F \mathbf{u} = 0$. What is Fundmental matrix in terms of K, K', R and t.

$$\mathbf{u}'^\top \mathbf{l}' = 0$$
$$\mathbf{u}'^\top (K'\mathbf{t} \times K' R K^{-1}\mathbf{u}) = 0$$
$$\mathbf{u}'^\top ([K'\mathbf{t}]_\times K' R K^{-1}\mathbf{u}) = 0$$
$$\mathbf{u}'^\top \boxed{\underbrace{[K'\mathbf{t}]_\times K' R K^{-1}}_{F}} \mathbf{u} = 0$$

$\Leftrightarrow K'(t_\times RK^{-1}) = F$

7. Define normalized coordinates as $\mathbf{p} = K^{-1}\mathbf{u}$ and $\mathbf{p}' = K'^{-1}\mathbf{u}'$. Repeat the above process for $\mathbf{p}$ and $\mathbf{p}'$.

$$\mathbf{p}' = R\mathbf{p} + \mathbf{t}$$
$$\mathbf{e}'_p = \mathbf{t}$$

Normal of the plane that contains both $\mathbf{p}$ and $\mathbf{e}'_p$,

$$\mathbf{n}' = \mathbf{e}'_p \times \mathbf{p}' = \mathbf{t} \times (R\mathbf{p} + \mathbf{t}) = \mathbf{t} \times R\mathbf{p}$$

Equation of the plane with normal $\mathbf{n}'$,

$$\mathbf{p}'^\top \mathbf{n} = 0$$
$$\mathbf{p}'^\top [\mathbf{t}]_\times R\mathbf{p} = 0$$

8. Define Essential matrix E such that $\mathbf{p}'^\top E\mathbf{p} = 0$. What is Essential matrix in terms of R and t?

$$\mathbf{p}'^\top \underbrace{[\mathbf{t}]_\times R}_{E} \mathbf{p} = 0$$

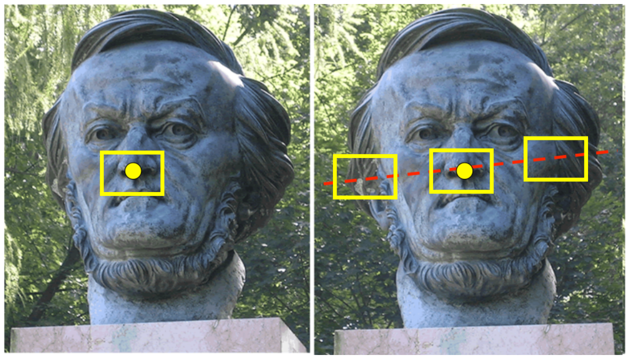
9. Find a relationship between Fundamental matrix and Essential matrix.

$$\mathbf{p}'^\top E\mathbf{p} = 0$$
$$\mathbf{u}'^\top \boxed{\underbrace{K'^{-\top} E K^{-1}}_{F}} \mathbf{u} = 0$$

K, K'

λ, λ' = unknown

R, t



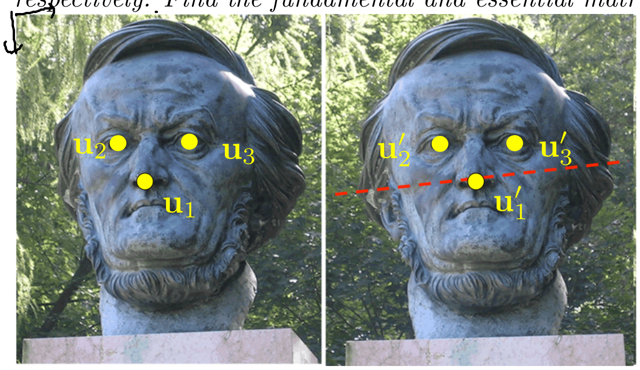
Matching

H, P

Problem 2 *Let $n \geq 8$ pairs of correspondence points be given between left image and the right images as $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ and $\mathbf{u}'_1, \mathbf{u}'_2, \ldots, \mathbf{u}'_n$ respectively. Let the left and right camera matrices be given as K and K' respectively. Find the fundamental and essential matrix.*

(0,0)

y



$$u_1 = \begin{bmatrix} 105 \\ 90 \\ 1 \end{bmatrix}$$

for all $i = \{1, \ldots, n\}$

$$u_i'^T F u_i = 0$$

$$(K'^{-1} u_i')^T E (K^{-1} u_i) = 0$$

1. Convert equations $\mathbf{u}_i'^{\top} F \mathbf{u}_i = 0$ for all $i \in \{1, \ldots, n\}$ into $A\mathbf{y} = 0$ form to find the unknown matrix F.

$$u_i'^T F u_i = 0$$

Is this equation linear?

$$\begin{bmatrix} x_i' & y_i' & w_i' \end{bmatrix} \begin{bmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{bmatrix} \begin{bmatrix} x_i \\ y_i \\ w_i \end{bmatrix} = 0$$

$$x_i' f_{11} x_i + x_i' f_{12} y_i + \ldots = 0$$

no quadratic or higher order terms

2. Write $F = \begin{bmatrix} \mathbf{f}_1^{\top} \\ \mathbf{f}_2^{\top} \\ \mathbf{f}_3^{\top} \end{bmatrix}$ in terms of its rows.

$$u_i'^T F u_i = 0 \Rightarrow u_i'^T \underbrace{\begin{bmatrix} f_1^T \\ f_2^T \\ f_3^T \end{bmatrix}}_{} u_i = 0$$

$$Ay = 0$$

$$u_i'^T \begin{bmatrix} f_1^T u_i \\ f_2^T u_i \\ f_3^T u_i \end{bmatrix} = 0$$

3. Expand $\mathbf{u}_i'^\top F \mathbf{u}_i = 0$ in terms of the rows of F.

$$u_i'^T \underbrace{\begin{bmatrix} f_1^T \\ f_2^T \\ f_3^T \end{bmatrix}}\, u_i = 0$$

4. Rearrange terms so that we can get the unknowns on the right hand side, so that we have an equation in the form $A_i\mathbf{y} = 0$ for a single point.

$f_1 \in \mathbb{R}^{3\times 1}$

f_2

f_3

$$u_i'^T \begin{bmatrix} u_i^T f_1 \\ u_i^T f_2 \\ u_i^T f_3 \end{bmatrix}_{3\times 1} = 0 \Rightarrow \underbrace{u_{i\,1\times 3}'^T \begin{bmatrix} u_i^T & 0_{3\times 1}^T & 0_{3\times 1}^T \\ 0_{3\times 1}^T & u_i^T & 0_{3\times 1}^T \\ 0^T & 0^T & u_i^T \end{bmatrix}_{3\times 9}} \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix}_{9\times 1} = 0$$

$\leftarrow 3 \rightarrow \leftarrow 3 \rightarrow \leftarrow 3 \rightarrow$

5. Concatenate equations from multiple points to a big matrix and equation of the form $A\mathbf{y} = 0$.

$$\underbrace{[x_i', y_i', w_i']}_{u_i'^T} \begin{bmatrix} u_i^T & 0^T & 0^T \\ 0^T & u_i^T & 0^T \\ 0^T & 0^T & u_i^T \end{bmatrix}$$

$\overset{3}{\leftrightarrow}$

$$\begin{bmatrix} x_i' u_i^T & y_i' u_i^T & w_i' u_i^T \end{bmatrix}_{1\times 9} \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix}_{9\times 1} = 0_{1\times 1}$$

$F = 2F$

$= 3F$

6. Solve the above equation using Singular value decomposition.

7. Find essential matrix from Fundamental matrix

$$F = K'^{-T} E K^{-1}$$

Left multiply by K'^T

$$K'^T F = (K'^T K'^{-T}) E K^{-1}$$

$$K'^T F = E K^{-1}$$

Right multiply by $\underline{K}$

$$K'^T F K = E (K^{-1} K)$$

$$K'^T F K = E$$

Problem 3 *Find the Rotation R and translation $\mathbf{t}$ from the Essential matrix $E = \underbrace{\boxed{\mathbf{t} \times R}}$.*

$$t_\times = \begin{bmatrix} 0 & -t_1 & t_2 \\ t_1 & 0 & -t_3 \\ -t_2 & t_3 & 0 \end{bmatrix}$$

R - orthonormal

$$
\begin{array}{cc}
 & \overbrace{\overset{3}{\longleftrightarrow}\quad\overset{3}{\longleftrightarrow}\quad\overset{3}{\longleftrightarrow}}^{}\\
 & \underset{9}{\longleftrightarrow} \\
n\Big\downarrow & \underbrace{\begin{bmatrix} x_1' u_1^T & y_1' u_1^T & w_1' u_1^T \\ x_2' u_2^T & y_2' u_2^T & w_2' u_2^T \\ \vdots & & \\ x_n' u_n^T & y_n' u_n^T & w_n u_n^T \end{bmatrix}_{n\times 9}}_{A}
\end{array}
\begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix}_{9\times 1} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \vdots \end{bmatrix}_{n\times 1}
$$

$$Ay = 0$$

$$\text{rank}(A) = 8 \checkmark$$

$$Ay = 0 \Rightarrow y \in \mathcal{N}(A)$$

$$A = U_{n\times n} \Sigma_{n\times 9} V^T_{9\times 9}$$

$$\Sigma = \begin{bmatrix} \sigma_1 & & 0 \\ & \ddots \; \sigma_8 & \\ 0 & & \end{bmatrix}_{n\times 9}$$

$$\begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} = \vec{v}_9 \in \mathbb{R}^{9\times 1}$$

$$V = \begin{bmatrix} \vec{v}_1 & \cdots & \vec{v}_9 \end{bmatrix}_{9\times 9}$$

$$F = \begin{bmatrix} f_1^T \\ f_2^T \\ f_3^T \end{bmatrix}$$

Solution

1. $$Z = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$[t]_\times = UZU^\top$

2. $$W = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \boxed{\mathrm{diag}(1,1,0)Z^\top}$$

3. Let the SVD of E be
$$E = U\Sigma V^\top$$

4. $$E = \underbrace{\underbrace{(UZU^\top)}_{[\mathbf{t}]_\times}}_{\lambda[t]_\times}\underbrace{\underbrace{(UW^\top V^\top)}_{R}}_{R}$$

Things to take away

1. Epipolar line can be found by projecting two points from left camera to the right camera. One of the points can be origin.
2. Definition of Fundamental matrix and Essenstial matrix.
3. How to find essential matrix from given correspondence points?
4. How to find Rotation and translation from essential matrix?